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HYPONORMALITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE OF AN ANNULUS

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Abstract. A bounded operator S on a Hilbert space is hyponormal if $S^*S - SS^*$ is positive. In this work we find necessary conditions for the hyponormality of the Toeplitz operator $T_{f+\overline{g}}$ on the Bergman space of the annulus $\{1/2 < |z| < 1\}$, where f and g are analytic and f satisfies a smoothness condition.

1. Introduction

A bounded operator S on a Hilbert space is hyponormal if $S^*S - SS^*$ is positive. Hyponormality of Toeplitz operators has been studied by many authors. Hyponormality of these operators on the Hardy space was considered in [3, 4]. Hyponormality of these operators with a symbol of the form $g_1 + \overline{g_2}$ on the Bergman space of the unit disk was first considered in [8]. Therein a necessary condition was proved, which was later improved in [1]. Some special cases are treated in [7]. A sufficient condition when g_1 is a monomial and g_2 is a polynomial is proved in [9]. An improvement of the necessary condition in the case when g_1 and g_2 are binomials is given in [5]. Basic material on Toeplitz operators on the Bergman space of the unit disk can be found in [2]. In this work we consider hyponormality of Toeplitz operators on the Bergman space of an annulus.

We start with definitions and notations. Denote by $A^2_{1/2}$ the space of holomorphic functions on the annulus $C_{1/2} = \{z \in \mathbb{C} : 1/2 < |z| < 1\}$ such that $\int |h|^2 \, dm(z) < \infty$, where $dm(z) = (4/3\pi) d\lambda(z)$ and λ is the Lebesgue measure on the annulus. If $h \in A^2_{1/2}$ we write $h = a_0 + \sum_1^\infty a_n z^n + a_{-n} z^{-n}$ and we have $\|h\|^2 = \sum_0^\infty \frac{4(1-(1/2)^{2n+2})}{3(n+1)} |a_n|^2 + \frac{8}{3} \ln 2 \, |a_{-1}|^2 + \sum_2^\infty \frac{4(2^{2n-2}-1)}{3(n-1)} |a_{-n}|^2$. We denote by $L^2(C_{1/2})$ the space of measurable and square integrable functions with respect to dm on $C_{1/2}$. Toeplitz operators on $A^2_{1/2}$ are defined by $T_f(h) = P(hf)$, where f is bounded and measurable on $C_{1/2}$, P is the orthogonal projection on

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$A^2_{1/2}$, and h is in $A^2_{1/2}$. The Hankel operators on the space $A^2_{1/2}$ are defined by $H_f(h) = (I-P)(hf)$. The space $A^2_{1/2}$ has an orthonormal basis given by the union of the sets

$$\left\{e_n = \frac{\sqrt{3(n+1)}}{2\sqrt{(1-(1/2)^{2n+2}}} z^n,\ n \geq 0\right\},$$
$$\left\{e_{-1} = \frac{\sqrt{3}}{\sqrt{8\ln 2z}}\right\}, \quad \text{and}$$
$$\left\{e_{-n} = \frac{\sqrt{3(n-1)}}{2\sqrt{(2^{2n-2}-1)}} \frac{1}{z^n},\ n \geq 2.\right\}.$$

We consider hyponormality of Toeplitz operators with a symbol of the form $f = g_1 + \overline{g_2}$, where g_1 and g_2 are bounded analytic functions on $C_{1/2}$. We begin by recalling some known properties of Toeplitz operators.

2. Some basic properties

Lemma 2.1. *Let f and g be bounded and measurable on $C_{1/2}$. The following properties hold:*

a) $T_{f+g} = T_f + T_g$.

b) $T_f^* = T_{\overline{f}}$.

c) $T_f T_g = T_{fg}$ *if g is analytic on $C_{1/2}$ or f is conjugate analytic.*

d) $T_{\overline{f}} T_f - T_f T_{\overline{f}} = H^*_{\overline{f}} H_{\overline{f}}$ *if f is analytic.*

The next proposition is easy to prove and its proof is omitted.

Proposition 2.2. *Let g_1 and g_2 be polynomials. The following are equivalent:*

a) $T_{g_1+\overline{g_2}}$ *is hyponormal.*

b) $T_{\overline{g_2}} T_{g_2} - T_{g_2} T_{\overline{g_2}} \leq T_{\overline{g_1}} T_{g_1} - T_{g_1} T_{\overline{g_1}}$.

c) $H^*_{\overline{g_2}} H_{\overline{g_2}} \leq H^*_{\overline{g_1}} H_{\overline{g_1}}$.

d) $H_{g_2} = K H_{g_1}$, *where K is an operator of norm less than one.*

The following lemma provides computations that will be needed.

Lemma 2.3. *The projection P on $A^2_{1/2}$ satisfies the following relations:*

1) $$P(z^m \overline{z^n}) = \frac{(m-n+1)(1-(1/2)^{2m+2})}{(m+1)(1-(1/2)^{2m-2n+2})} z^{m-n}, \text{ if } m \geq n.$$

2) $$P(z^m \overline{z^n}) = \frac{(n-m-1)(1-(1/2)^{2m+2)})}{(m+1)(2^{2n-2m-2}-1)} \frac{1}{z^{n-m}}, \text{ if } n \geq m+2.$$

3) $$P(z^m \overline{z^{m+1}}) = \frac{(1-(1/2)^{2m+2})}{2\ln 2(m+1)} \frac{1}{z}, \text{ if } n = m+1.$$

4) $$P\left(\frac{1}{z^m} \overline{z^n}\right) = \frac{(m+n-1)(2^{2m-2}-1))}{(2^{2(m+n)-2}-1)(m-1)} \frac{1}{z^{m+n}}, \text{ if } m \geq 2.$$

5) $P\left(\frac{1}{z}\overline{z^n}\right) = \frac{2n\ln 2}{(2^{2n}-1)}\frac{1}{z^{n+1}}$, *if* $n \geq 1$.

6) $P\left(\frac{1}{\overline{z^m}}z^n\right) = \frac{(m+n+1)((1-(1/2)^{2n+2})}{(n+1)(1-(1/2)^{2(m+n)+2})}z^{m+n}$.

7) $P\left(\frac{1}{\overline{z^m}z^n}\right) = \frac{((m-n)+1)(2^{2n-2}-1)}{(n-1)(1-(1/2)^{2(m-n)+2})}z^{m-n}$, *if* $m \geq n$, $n \neq 1$.

8) $P\left(\frac{1}{\overline{z^m}z}\right) = \frac{2m\ln 2}{(1-(1/2)^{2m})}z^{m-1}$, *if* $m \geq 1$.

9) $P\left(\frac{1}{\overline{z^m}z^n}\right) = \frac{(n-m-1)(2^{2n-2}-1)}{(n-1)(2^{2(n-m)-2}-1)}\frac{1}{z^{n-m}}$, *if* $m \geq 1$, $n-m > 1$.

10) $P\left(\frac{1}{\overline{z^m}z^{m+1}}\right) = \frac{(2^{2m}-1)}{2m\ln 2}\frac{1}{z}$, *if* $m \geq 1$.

3. First main result

We begin with a matrix computation.

Lemma 3.1. *Let* $f = \sum_1^\infty a_k z^k$ *be bounded on* $C_{1/2}$. *Then for* $i, j \geq 1$ *we have*

$$\begin{aligned}
&\left\langle T_{\overline{f}}T_f - T_f T_{\overline{f}}(e_j), e_i\right\rangle \\
&= \sum_{\substack{1\leq k\\ 1\leq k+j-i}} \overline{a_{k+j-i}}a_k \frac{\sqrt{i+1}\sqrt{j+1}(1-(1/2)^{2(k+j)+2)})}{\sqrt{1-(1/2)^{2i+2}}\sqrt{1-(1/2)^{2j+2}}(k+j+1)} \\
&\quad - \sum_{\substack{1\leq k\leq j\\ 1\leq k+i-j}} \overline{a_k}a_{k+i-j}\frac{(j-k+1)\sqrt{1-(1/2)^{2i+2}}\sqrt{1-(1/2)^{2j+2}}}{(1-(1/2)^{2(j-k)+2})\sqrt{i+1}\sqrt{j+1}} \\
&\quad - \overline{a_{j+1}}a_{i+1}\frac{\sqrt{(1-(1/2)^{2i+2}}\sqrt{(1-(1/2)^{2j+2}}}{2\ln 2\sqrt{i+1}\sqrt{j+1}} \\
&\quad - \sum_{\substack{j+2\leq k\\ 1\leq k+i-j}} \overline{a_k}a_{k+i-j}\frac{(k-i-1)\sqrt{(1-(1/2)^{2i+2}}\sqrt{(1-(1/2)^{2j+2}}}{\sqrt{i+1}\sqrt{j+1}}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\left\langle T_{\overline{f}}T_f(e_j), e_i\right\rangle &= \sum_{k,l=1}^\infty \overline{a_l}a_k \frac{\sqrt{3(i+1)}}{2\sqrt{(1-(1/2)^{2i+2}}}\frac{\sqrt{3(j+1)}}{2\sqrt{(1-(1/2)^{2j+2}}}\left\langle z^{k+j}, z^{i+l}\right\rangle \\
&= \sum_{\substack{1\leq k\\ 1\leq k+j-i}} \frac{\overline{a_{k+j-i}}a_k(1-(1/2)^{2(k+j)+2)})\sqrt{(i+1)(j+1)}}{(k+j+1)\sqrt{(1-(1/2)^{2i+2})(1-(1/2)^{2j+2})}}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}\left\langle T_f T_{\overline{f}}(e_j), e_i\right\rangle = &\sum_{\substack{1\le k+i-j\\1\le k\le j}} \frac{\overline{a_k} a_{k+i-j}(j-k+1)\sqrt{1-(1/2)^{2i+2}}\sqrt{1-(1/2)^{2j+2}}}{(1-(1/2)^{2(j-k)+2})\sqrt{i+1}\sqrt{j+1}}\\ &+\overline{a_{j+1}}a_{i+1}\frac{\sqrt{(1-(1/2)^{2i+2}}}{2\ln 2\sqrt{i+1}}\,\frac{\sqrt{(1-(1/2)^{2j+2}}}{\sqrt{j+1}}\\ &+\sum_{\substack{j+2\le k\\1\le k+i-j}} \frac{\overline{a_k} a_{k+i-j}(k-j-1)\sqrt{1\left(-(1/2)^{2i+2}\right)\left(1-(1/2)^{2j+2}\right)}}{\sqrt{(i+1)\,(j+1)}}.\end{aligned}$$

□

Set $\beta_{i,j} = \left\langle T_{\overline{f}}T_f - T_f T_{\overline{f}}(e_j), e_i\right\rangle$, $i, j \ge 1$. By rewriting the expression for $\beta_{i,j}$ we obtain

$$\begin{aligned}\beta_{i+p,i} = &\sum_{\substack{1\le k\le i\\1\le k+p}} \overline{a_k} a_{k+p}\frac{\sqrt{i+1}\sqrt{i+p+1}(1-(1/2)^{2(k+p+i)+2)})}{\sqrt{1-(1/2)^{2i+2}}\sqrt{1-(1/2)^{2(i+p)+2}}(k+p+i+1)}\\ &-\sum_{\substack{1\le k\le i\\1\le k+p}} \overline{a_k} a_{k+p}\frac{(i-k+1)\sqrt{1-(1/2)^{2i+2}}\sqrt{1-(1/2)^{2(i+p)+2}}}{(1-(1/2)^{2(i-k)+2})\sqrt{i+1}\sqrt{i+p+1}}\\ &+\overline{a_{i+1}}a_{i+p+1}\frac{\sqrt{i+1}\sqrt{i+p+1}(1-(1/2)^{2(2i+1+p)+2)})}{\sqrt{1-(1/2)^{2i+2}}\sqrt{1-(1/2)^{2(i+p)+2}}(2(i+1)+p)}\\ &-\overline{a_{i+1}}a_{i+p+1}\frac{\sqrt{(1-(1/2)^{2i+2}}\sqrt{(1-(1/2)^{2(i+p)+2}}}{2\ln 2\sqrt{i+1}\sqrt{i+p+1}}\\ &+\sum_{i+2\le k} \overline{a_k} a_{k+p}\frac{\sqrt{i+1}\sqrt{i+p+1}(1-(1/2)^{2(k+p+i)+2)})}{\sqrt{1-(1/2)^{2i+2}}\sqrt{1-(1/2)^{2(i+p)+2}}(k+p+i+1)}\\ &-\sum_{i+2\le k} \overline{a_k} a_{k+p}\frac{(k-i-1)\sqrt{(1-(1/2)^{2i+2}}\sqrt{(1-(1/2)^{2(i+p)+2}}}{\sqrt{i+1}\sqrt{i+p+1}}\\ = &\sum_{\substack{1\le k\le i\\1\le k+p}} \overline{a_k} a_{k+p} Q_{i,k,p} + \overline{a_{i+1}}a_{i+p+1}R_{i,p} + \sum_{i+2\le k}\overline{a_k}a_{k+p}S_{i,k,p}.\end{aligned}$$

Lemma 3.2. *We have* $\lim_{i\to\infty} i^2\beta_{i+p,i} = \gamma_{i+p,i}$*, where* $(\gamma_{i,j})$ *is the matrix of the Hardy space Topelitz operator* $T_{|f'|^2}$*.*

Proof. An elementary computation shows that $\lim_{i\to\infty} i^2 Q_{i,k,p} = k(k+p)$. Set $h_i(k) = i^2\chi_{\{1,\dots,i\}}(k)\,\overline{a_k}a_{k+p}Q_{i,k,p}$. The first sum in the above expression of $\beta_{i+p,i}$ can be written as $\int h_i(k)\,d\mu(k)$, where $d\mu$ is the counting measure. It is easy to see that for i sufficiently large, $|h_i(k)| \le 2|a_k a_{k+p}| \le k^2|a_k|^2 + (k+p)^2|a_{k+p}|^2 = M(k)$. Since $f' \in H^2$, the function $M(k)$ is integrable with respect to the counting measure.

By the dominated convergence theorem we obtain:

$$\lim_{i\to\infty} i^2 \sum_{\substack{1\le k\le i\\ 1\le k+p}} \overline{a_k} a_{k+p} Q_{i,k,p} = \sum k(k+p)\overline{a_k} a_{k+p}.$$

Also, for i large, there exists a constant C such that

$$|i^2\, \overline{a_{i+1}} a_{i+p+1} R_{i,p}| \le C\left((i+1)^2|a_{i+1}|^2 + (i+p+1)^2|a_{i+p+1}|^2\right).$$

Thus $\lim_{i\to\infty} i^2\, \overline{a_{i+1}} a_{i+p+1} R_{i,p} = 0$. Finally, it is not difficult to see that $i^2|S_{i,k,p}| \le k(k+p)$. Using the dominated convergence theorem we obtain

$$\lim_{i\to\infty} i^2 \sum_{i+2\le k} \overline{a_k} a_{k+p} S_{i,k,p} = 0.$$

We deduce that $\lim_{i\to\infty} i^2\beta_{i+p,i} = \sum k(k+p)\overline{a_k} a_{k+p}$ and recognize this last limit as being equal to $\gamma_{i+p,i}$, where $(\gamma_{i,j})$ is the matrix of the Hardy space Toeplitz operator $T_{|f'|^2}$. ☐

We are led to the following necessary condition for hyponormality.

Theorem 3.3. *Let $f = \sum_1^\infty a_k z^k$ and $g = \sum_1^\infty b_k z^k$ be bounded on $C_{1/2}$. Assume that $f' \in H^2$. If $T_{f+\overline{g}}$ is hyponormal then $g' \in H^2$ and $|g'| \le |f'|$ a.e. on the unit circle.*

Proof. If $(\theta_{i,j})$ denotes the matrix of $T_{\overline{f}} T_f - T_f T_{\overline{f}} - T_{\overline{g}} T_g - T_g T_{\overline{g}}$ and $(\sigma_{i,j})$ denotes the matrix of $T_{\overline{g}} T_g - T_g T_{\overline{g}}$, then the inequality $\sigma_{i,i} \le \beta_{i,i}$ leads to

$$\sum_{1\le k\le i} |b_k|^2 Q_{i,k,0} + |b_{i+1}|^2 R_{i,0} + \sum_{i+2\le k} |b_k|^2 S_{i,k,0}$$
$$\le \sum_{1\le k\le i} |a_k|^2 Q_{i,k,0} + |a_{i+1}|^2 R_{i,0} + \sum_{i+2\le k} |a_k|^2 S_{i,k,0}.$$

We deduce that $\sum_{1\le k\le i,} i^2|b_k|^2 Q_{i,k,0} \le i^2\beta_{i,i}$. Since $\lim_{i\to\infty} i^2 Q_{i,k,0} = k^2$, writing the left hand side of this last inequality as an integral with respect to the counting measure and using Fatou's lemma we get $\sum k^2|b_k|^2 \le \sum k^2|a_k|$ and $g' \in H^2$. From the previous lemma, $\lim_{i\to\infty} i^2\,\theta_{i+p,i} = \lambda_{i+p,i}$, where $(\lambda_{i,j})$ denotes the matrix of the Hardy space Toeplitz operator $T_{|f'|^2 - |g'|^2}$. Hyponormality and a property of Toeplitz matrices [6] lead to $|g'| \le |f'|$ a.e. on the unit circle. ☐

Corollary 3.4. *Let $f = \sum_1^\infty a_k z^k$ and $g = \sum_1^\infty b_k z^k$ be analytic and univalent in an open set containing $C_{1/2}$. Then $T_{f+\overline{g}}$ is normal if and only if $g = cf$, where c is a constant with $|c| = 1$.*

Proof. Only the necessary condition needs to be shown. Normality implies that $|g'| = |f'|$ on the unit circle. Thus f' and g' have the same finite number of zeros (if any) with the same multiplicity. We thus have $\frac{|f'|}{|g'|} = \frac{|g'|}{|f'|} = 1$ on the unit circle. By the maximum principle, $g' = cf'$ with $|c| = 1$. We get $g = cf$. ☐

Lemma 3.5. *Let $f=\sum_1^\infty a_k z^k$ be bounded on $C_{1/2}$. Then for $i\geq 3$, $j\geq 3$ we have*

$$\begin{aligned}
&\left\langle T_{\overline{f}}T_f - T_fT_{\overline{f}}(e_{-j}), e_{-i}\right\rangle \\
&\quad = \sum_{\substack{1\leq k<j-1\\ 1\leq k+i-j}} \overline{a_{k+i-j}}a_k \frac{\sqrt{(i-1)}}{\sqrt{(2^{2i-2}-1)}}\frac{\sqrt{(j-1)}}{\sqrt{(2^{2j-2}-1)}}\frac{(2^{2(j-k)-2}-1)}{(j-k-1)} \\
&\qquad + 2\ln 2\, \overline{a_{i-1}}a_{j-1}\frac{\sqrt{i-1}}{\sqrt{2^{2i-2}-1}}\frac{\sqrt{j-1}}{\sqrt{2^{2j-2}-1}} \\
&\qquad + \sum_{j\leq k}\overline{a_{k+i-j}}a_k \frac{\sqrt{(i-1)}}{\sqrt{(2^{2i-2}-1)}}\frac{\sqrt{(j-1)}}{\sqrt{(2^{2j-2}-1)}}\frac{(1-(1/2)^{2(k-j)+2})}{k-j+1} \\
&\qquad - \sum_{\substack{1\leq k\\ 1\leq k+j-i}}\overline{a_k}a_{k+j-i}\frac{(k+j-1)\sqrt{(2^{2i-2}-1)}\sqrt{2^{2j-2}-1}}{(2^{2(j+k)-2}-1)\sqrt{i-1}\sqrt{j-1}}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\left\langle T_{\overline{f}}T_f(e_{-j}), e_{-i}\right\rangle &= \sum_{\substack{1\leq k<j-1\\ 1\leq k+i-j}} \overline{a_{k+i-j}}a_k \frac{\sqrt{i-1}}{\sqrt{2^{2i-2}-1}}\frac{\sqrt{j-1}}{\sqrt{2^{2j-2}-1}}\frac{(2^{2(j-k)-2}-1)}{(j-k-1)} \\
&\quad + 2\ln 2\, \overline{a_{i-1}}a_{j-1}\frac{\sqrt{i-1}}{\sqrt{2^{2i-2}-1}}\frac{\sqrt{j-1}}{\sqrt{2^{2j-2}-1}} \\
&\quad + \sum_{j\leq k}\overline{a_{k+i-j}}a_k \frac{\sqrt{(i-1)}}{\sqrt{(2^{2i-2}-1)}}\frac{\sqrt{(j-1)}}{\sqrt{(2^{2j-2}-1)}}\frac{(1-(1/2)^{2(k-j)+2})}{k-j+1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left\langle T_fT_{\overline{f}}(e_{-j}), e_{-i}\right\rangle &= \sum_{k,l=1}^{\infty}\overline{a_k}a_l\frac{\sqrt{3(i-1)}}{2\sqrt{(2^{2i-2}-1)}}\frac{\sqrt{3(j-1)}}{2\sqrt{(2^{2j-2}-1)}}\left\langle P\left(\overline{z^k}\frac{1}{z^j}\right), P(\overline{z^l}\frac{1}{z^i})\right\rangle \\
&= \sum_{\substack{1\leq k\\ 1\leq k+j-i}}\overline{a_k}a_{k+j-i}\frac{(k+j-1)\sqrt{2^{2i-2}-1}\sqrt{2^{2j-2}-1}}{(2^{2(j+k)-2}-1)\sqrt{i-1}\sqrt{j-1}}.
\end{aligned}$$

□

Let $\beta_{-i,-j}=\left\langle (T_{\overline{f}}T_f - T_fT_{\overline{f}})(e_{-j}), e_{-i}\right\rangle$ and denote by $(\zeta_{i,j})$ the matrix of the Toeplitz operator $T_{|f'_{1/2}|^2}$ on the Hardy space of the unit disk, where $f_{1/2}(z)=\sum \overline{a_k}\frac{z^k}{2^k}$.

We can show the following lemma.

Lemma 3.6. *We have* $\lim_{i\to\infty} i^2\beta_{-i-p,-i} = \zeta_{i+p,i}$.

Proof.

$$
\begin{aligned}
&\beta_{-i-p,-i}\\
&= \sum_{\substack{1\le k<i-1\\1\le k+p}} \overline{a_{k+p}}a_k \frac{\sqrt{(i-1)}}{\sqrt{(2^{2i-2}-1)}}\frac{\sqrt{(i+p-1)}}{\sqrt{(2^{2(i+p)-2}-1)}}\frac{(2^{2(i-k)-2}-1)}{(i-k-1)}\\
&\quad + 2\ln 2\overline{a_{i+p-1}}a_{i-1}\frac{\sqrt{i+p-1}}{\sqrt{2^{2(i+p)-2}-1}}\frac{\sqrt{i-1}}{\sqrt{2^{2i-2}-1}}\\
&\quad + \sum_{i\le k}\overline{a_{k+p}}a_k\frac{\sqrt{i-1}}{\sqrt{(2^{2i-2}-1)}}\frac{\sqrt{i+p-1}}{\sqrt{(2^{2(i+p)-2}-1)}}\frac{(1-(1/2)^{2(k-i)+2})}{k-i+1}\\
&\quad - \sum_{\substack{1\le k\\1\le k+p}}\overline{a_{k+p}}a_k\frac{(k+p+i-1)\sqrt{(2^{2i-2}-1)}\sqrt{2^{2(i+p)-2}-1}}{(2^{2(i+k+p)-2}-1)\sqrt{i-1}\sqrt{i+p-1}}\\
&= \sum_{\substack{1\le k<i-1\\1\le k+p}}\frac{\overline{a_{k+p}}a_k(i-1)(i+p-1)(2^{2(i-k)-2}-1)(2^{2(i+k+p)-2}-1)}{\sqrt{(2^{2i-2}-1)(2^{2(i+p)-2}-1)}\sqrt{(i-1)(i+p-1)}(i-k-1)(2^{2(i+k+p)-2}-1)}\\
&\quad - \sum_{\substack{1\le k<i-1\\1\le k+p}}\frac{\overline{a_{k+p}}a_k(k+p+i-1)(i-k-1)(2^{2i-2}-1)(2^{2(i+p)-2}-1)}{\sqrt{(2^{2i-2}-1)(2^{2(i+p)-2}-1)}\sqrt{(i-1)(i+p-1)}(i-k-1)(2^{2(i+k+p)-2}-1)}\\
&\quad + \overline{a_{i+p-1}}a_{i-1}\left(2\ln 2\frac{\sqrt{i-1}}{\sqrt{2^{2i-2}-1}}\frac{\sqrt{i+p-1}}{\sqrt{2^{2(i+p)-2}-1}}\right.\\
&\qquad\qquad \left. - \frac{(2i-2+p)\sqrt{(2^{2i-2}-1)}\sqrt{2^{2(i+p)-2}-1}}{(2^{2(2i-1+p)-2}-1)\sqrt{i-1}\sqrt{i+p-1}}\right)\\
&\quad + \sum_{i\le k}\overline{a_{k+p}}a_k\left(\frac{\sqrt{i-1}}{\sqrt{(2^{2i-2}-1)}}\frac{\sqrt{i+p-1}}{\sqrt{(2^{2(i+p)-2}-1)}}\frac{(1-(1/2)^{2(k-i)+2})}{k-i+1}\right)\\
&\quad - \sum_{i\le k}\overline{a_{k+p}}a_k\left(\frac{(k+p+i-1)\sqrt{(2^{2i-2}-1)}\sqrt{2^{2(i+p)-2}-1}}{(2^{2(i+k+p)-2}-1)\sqrt{i-1}\sqrt{i+p-1}}\right)\\
&= \sum_{\substack{1\le k<i-1\\1\le k+p}}\overline{a_{k+p}}a_kQ'_{i,p,k} + \overline{a_{i+p-1}}a_{i-1}R'_{i,p} + \sum_{i\le k}\overline{a_{k+p}}a_kS'_{i,k,p}.
\end{aligned}
$$

A computation shows that $\lim\limits_{i\to\infty} i^2Q'_{i,p,k} = \frac{1}{2^{2k+p}}$. As in the proof of the previous theorem we can show that

$$\lim_{i\to\infty} i^2 \sum_{\substack{1\le k<i-1\\1\le k+p}}\overline{a_{k+p}}a_kQ'_{i,p,k} = \sum_{\substack{1\le k\\1\le k+p}} k(k+p)\frac{a_k}{2^k}\frac{\overline{a_{k+p}}}{2^{k+p}}.$$

We see that this last limit is equal to $\zeta_{i,i+p}$. We also show that

$$\lim_{i\to\infty} i^2\,\overline{a_{i+p-1}}a_{i-1}R'_{i,p} = 0$$

and

$$\lim_{i\to\infty} i^2 \sum_{i\le k} \overline{a_{k+p}} a_k S'_{i,k,p} = 0.$$

We deduce that

$$\lim_{i\to\infty} i^2 \beta_{-i-p,-i} = \zeta_{i+p,i}.$$

If $f = \sum_1^\infty a_k z^k$ is bounded analytic on $C_{1/2}$, then clearly $\sum \frac{k^2}{2^{2k}} |a_k|^2 < \infty$. We can also see that $|g'_{1/2}| \le |f'_{1/2}|$ a.e. on the unit circle is equivalent to $|g'| \le |f'|$ a.e. on $\{z : |z| = 1/2\}$. □

Theorem 3.7. *Let $f = \sum_1^\infty a_k z^k$ and $g = \sum_1^\infty b_k z^k$ be bounded on $C_{1/2}$. If $T_{f+\overline{g}}$ is hyponormal then $|g'| \le |f'|$ a.e. on $\{z : |z| = 1/2\}$.*

The proof is similar to the proof of the previous theorem and is omitted. Combining the previous two theorems we get our first main result.

Theorem 3.8. *Let $f = \sum_1^\infty a_k z^k$ and $g = \sum_1^\infty b_k z^k$ be bounded on $C_{1/2}$ and assume that $f' \in H^2$. If $T_{f+\overline{g}}$ is hyponormal then $g' \in H^2$ and $|g'| \le |f'|$ a.e. on $\{z : |z| = 1\} \cup \{z : |z| = 1/2\}$.*

4. Second main result

We now put $f = \sum_1^\infty a_k \frac{1}{z^k}$ and $g = \sum_1^\infty b_k \frac{1}{z^k}$ and assume that f and g are bounded on $C_{1/2}$. We need the following computation.

Lemma 4.1. *For $i \ge 1$, $j \ge 1$ we have*

$$\begin{aligned}&\left\langle T_{\overline{f}} T_f - T_f T_{\overline{f}}(e_j), e_i \right\rangle \\ &\quad= \sum_{1\le k, k+i-j} \overline{a_{k+i-j}} a_k \frac{\sqrt{(i+1)}\sqrt{(j+1)}(1-(1/2)^{2(j-k)+2})}{\sqrt{(1-(1/2)^{2i+2}}\sqrt{(1-(1/2)^{2j+2}}(j-k+1)} \\ &\qquad - \sum_{1\le k, k+j-i} \overline{a_k} a_{k+j-i} \frac{\sqrt{(1-(1/2)^{2i+2})}\sqrt{(1-(1/2)^{2j+2}}(j+k+1)}{\sqrt{i+1}\sqrt{j+1}(1-(1/2)^{2(j+k)+2})}.\end{aligned}$$

Proof. We have

$$\begin{aligned}\left\langle T_{\overline{f}} T_f(e_j), e_i \right\rangle &= \sum_{k,l=1}^{\infty} \overline{a_l} a_k \frac{\sqrt{3(i+1)}}{2\sqrt{(1-(1/2)^{2i+2}}} \frac{\sqrt{3(j+1)}}{2\sqrt{(1-(1/2)^{2j+2}}} \left\langle z^{j-k}, z^{i-l} \right\rangle \\ &= \sum_{1\le k, k+i-j}^{\infty} \overline{a_{k+i-j}} a_k \frac{\sqrt{(i+1)}}{\sqrt{(1-(1/2)^{2i+2}}} \frac{\sqrt{(j+1)}}{\sqrt{(1-(1/2)^{2j+2}}} \frac{(1-(1/2)^{2(j-k)+2}}{j-k+1}\end{aligned}$$

and

$$\begin{aligned}\langle T_f T_{\overline{f}}(e_j), e_i\rangle &= \sum_{k,l=1}^{\infty} \overline{a_k} a_l \frac{\sqrt{3(i+1)}}{2\sqrt{(1-(1/2)^{2i+2}}} \frac{\sqrt{3(j+1)}}{2\sqrt{(1-(1/2)^{2j+2}}} \\ &\quad \times \left\langle P\left(\frac{1}{\overline{z^k}} z^j\right), P\left(\frac{1}{\overline{z^l}} z^i\right)\right\rangle \\ &= \sum_{1\le k, k+j-i}^{\infty} \overline{a_k} a_{k+j-i} \frac{\sqrt{(1-(1/2)^{2i+2})}\sqrt{(1-(1/2)^{2j+2}}(j+k+1)}{\sqrt{i+1}\sqrt{j+1}(1-(1/2)^{2(j+k)+2})}.\end{aligned}$$

□

We get, using the same notations as before,

$$\begin{aligned}\beta_{i+p,i} &= \sum_{\substack{1\le k-p \\ 1\le k}} \overline{a_k} a_{k-p} \frac{\sqrt{(i+1)}\sqrt{(i+p+1)}(1-(1/2)^{2(i-k+p)+2})}{\sqrt{(1-(1/2)^{2i+2}}\sqrt{(1-(1/2)^{2(i+p)+2}} i-k+p+1} \\ &\quad - \sum_{\substack{1\le k \\ 1\le k-p}} \overline{a_k} a_{k-p} \frac{\sqrt{(1-(1/2)^{2i+2})}\sqrt{(1-(1/2)^{2(i+p)+2}}(i+k+1)}{\sqrt{i+1}\sqrt{i+p+1}(1-(1/2)^{2(i+k)+2})} \\ &= \sum_{\substack{1\le k \\ 1\le k-p}} \overline{a_k} a_{k-p} U_{i,k,p}.\end{aligned}$$

A computation shows that

$$\lim_{i\to\infty} i^2 \beta_{i+p,i} = \sum_{1\le k, k-p} \overline{a_k} a_{k-p} k(k-p).$$

We recognize the general element $\xi_{m+p,m}$ of the matrix of the Toeplitz operator $T_{|\widetilde{f'}|}$ on the Hardy space of the unit disk with $\widetilde{f}$ defined by $\widetilde{f(z)} = \sum_1^\infty \overline{a_k} z^k$. Obviously the condition $|\widetilde{g}'(e^{i\theta})| \le |\widetilde{f}'(e^{i\theta})|$ a.e. on the unit circle is the same as $|g'| \le |f'|$ a.e. on the unit circle. The condition $\widetilde{f}' \in H^2$ is equivalent to $\sum k^2|a_k|^2 < \infty$ and this is satisfied if $f = \sum_1^\infty a_k \frac{1}{z^k}$ is bounded on $C_{1/2}$. Using similar methods we obtain the following theorem.

Theorem 4.2. *Let $f = \sum_1^\infty a_k \frac{1}{z^k}$ and $g = \sum_1^\infty b_k \frac{1}{z^k}$ be analytic and bounded on $C_{1/2}$. If $T_{f+\overline{g}}$ is hyponormal then $|g'| \le |f'|$ a.e. on the unit circle.*

If we set $f_2(z) = \sum 2^k a_k z^k$, then $f_2' \in H^2$ is equivalent to $\sum k^2 2^{2k}|a_k|^2 < \infty$. In this case, $|g_2'| \le |f_2'|$ a.e. on the unit circle is equivalent to $|g'| \le |f'|$ a.e. on $\{z : |z| = 1/2\}$. Let $(\rho_{i,j})$ denote the matrix of the Hardy space Toeplitz operator $T_{|f_2'|^2}$. Using the same notations we can show the following lemma, the proof of which is omitted.

Lemma 4.3. $\lim_{i\to\infty} i^2 \beta_{-i-p,-i} = \rho_{i+p,i}$.

We obtain our second main result.

Theorem 4.4. *Let* $f = \sum_1^\infty a_k \frac{1}{z^k}$ *and* $g = \sum_1^\infty b_k \frac{1}{z^k}$ *be bounded on* $C_{1/2}$*, with* $\sum k^2 2^{2k} |a_k|^2 < \infty$*. If* $T_{f+\overline{g}}$ *is hyponormal then* $\sum k^2 2^{2k} |b_k|^2 < \infty$ *and* $|g'| \leq |f'|$ *a.e. on* $\{z : |z| = 1\} \cup \{z : |z| = 1/2\}$*.*

An application of the maximum modulus principle allows us to describe the normality of $T_{f+\overline{g}}$ under the condition of univalence.

Corollary 4.5. *Let* $f = \sum_1^\infty a_k \frac{1}{z^k}$ *and* $g = \sum_1^\infty b_k \frac{1}{z^k}$ *be analytic and univalent in an open set containing* $C_{1/2}$*. Then* $T_{f+\overline{g}}$ *is normal if and only if* $g = cf$*, where* c *is a constant with* $|c| = 1$*.*

We list two more results which are shown using methods similar to the ones used for the previous theorems.

Theorem 4.6. *Let* $f = \sum_1^\infty a_k z^k$ *and* $g = \sum_1^\infty b_k \frac{1}{z^k}$ *be bounded on* $C_{1/2}$*. Assume that* $\sum k^2 |a_k|^2 < \infty$*. If* $T_{f+\overline{g}}$ *is hyponormal then* $\sum k^2 |b_k|^2 < \infty$ *and* $|g'(e^{i\theta})| \leq |f'(e^{i\theta})|$ *a.e. on the unit circle.*

Corollary 4.7. *Let* $f = \sum_1^\infty a_k z^k$ *and* $g = \sum_1^\infty b_k \frac{1}{z^k}$ *be bounded on* $C_{1/2}$*. Assume that* f *and* $\widetilde{g}$ *are univalent in an open set containig* $C_{1/2}$*. Then* $T_{f+\overline{g}}$ *is normal if and only if* $\widetilde{g} = cf$ *for some constant* c *with* $|c| = 1$*.*

Theorem 4.8. *Let* $f = \sum_1^\infty a_k z^k$ *and* $g = \sum_1^\infty b_k \frac{1}{z^k}$ *be bounded on* $C_{1/2}$*. If* $T_{f+\overline{g}}$ *is hyponormal then* $\sum k^2 2^{2k} |b_k|^2 < \infty$ *and* $|g'(\frac{1}{2}e^{i\theta})| \leq |f'(\frac{1}{2}e^{i\theta})|$ *for almost all* θ*.*

Corollary 4.9. *Let* $f = \sum_1^\infty a_k z^k$ *and* $g = \sum_1^\infty b_k \frac{1}{z^k}$ *be bounded on* $C_{1/2}$ *and assume that* $T_{f+\overline{g}}$ *is hyponormal. The following holds:*

i) $\sum k^2 2^{2k} |b_k|^2 < \infty$ *and* $|g'(\frac{1}{2}e^{i\theta})| \leq |f'(\frac{1}{2}e^{i\theta})|$ *for almost all* θ*.*

ii) If $f' \in H^2$ *then* $|g'(e^{i\theta})| \leq |f'(e^{i\theta})|$ *a.e. on the unit circle.*

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