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ON JACOBSON'S LEMMA AND CLINE'S FORMULA FOR DRAZIN INVERSES

DIJANA MOSIĆ

Abstract. Under new conditions $bac = bdb$ and $cdb = cac$, we present extensions of Jacobson's lemma and Cline's formula for the generalized Drazin inverse and pseudo Drazin inverse in a ring. Applying these results, we give Jacobson's lemma for the Drazin inverse, group inverse, and ordinary inverse, and Cline's formula for the Drazin inverse.

1. Introduction

Let $\mathcal{R}$ be an associative ring with unit 1. We use $\mathcal{R}^{-1}$ and $\mathcal{R}^{\text{nil}}$ to denote the sets of all invertible and nilpotent elements of $\mathcal{R}$, respectively. An element $q \in \mathcal{R}$ is quasinilpotent if $1 + xq \in \mathcal{R}^{-1}$ for all $x \in \text{comm}(q)$, where $\text{comm}(q) = \{z \in \mathcal{R} : qz = zq\}$ is the commutant of q. The set of all quasinilpotent elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{\text{qnil}}$.

The concept of the generalized Drazin inverse in Banach algebras was introduced by Koliha [7]. Koliha and Patrício [8] extended this notion from Banach algebras to rings. The generalized Drazin inverse of an element $a \in \mathcal{R}$ is an element $x \in \mathcal{R}$ such that

$$x \in \text{comm}^2(a), \quad xax = x, \quad \text{and} \quad a(1 - ax) \in \mathcal{R}^{\text{qnil}},$$

where $\text{comm}^2(a) = \{z \in \mathcal{R} : zy = yz \text{ for all } y \in \text{comm}(a)\}$ is the double commutant of a. If the generalized Drazin inverse x of a exists, then it is unique [8] and is denoted by a^d. In Banach algebras it is enough to assume $x \in \text{comm}(a)$ instead of $x \in \text{comm}^2(a)$ in the definition of the generalized Drazin inverse. $\mathcal{R}^d$ will denote the set of all generalized Drazin invertible elements of $\mathcal{R}$.

Lemma 1.1 ([8, Theorem 4.2]). *Let $a \in \mathcal{R}$. Then $a \in \mathcal{R}^d$ if and only if there exists $p = p^2 \in \mathcal{R}$ such that*

$$p \in \text{comm}^2(a), \quad a + p \in \mathcal{R}^{-1}, \quad \text{and} \quad ap \in \mathcal{R}^{\text{qnil}}.$$

In this case, $p = 1 - aa^d$ is a spectral idempotent of a and will be denoted by a^π.

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In the case that $a(1-ax) \in \mathcal{R}^{\text{nil}}$ instead of $a(1-ax) \in \mathcal{R}^{\text{qnil}}$ in the definition of the generalized Drazin inverse, $a^d = a^D$ is the Drazin inverse of a [5]. The condition of $a - a^2x$ being nilpotent is equivalent to $a^{k+1}x = a^k$, for some non-negative integer k. The smallest such k is called the index of a and it is denoted by $\text{ind}(a)$. When $a = axa$ instead of $a - a^2x \in \mathcal{R}^{\text{nil}}$ in the definition of the Drazin inverse, $a^D = a^\#$ is the group inverse of a. The subsets of $\mathcal{R}$ composed of Drazin invertible and group invertible elements will be denoted by $\mathcal{R}^D$ and $\mathcal{R}^\#$, respectively. The concept of Drazin inverse plays an important role in various fields like Markov chains, singular differential and difference equations, iterative methods, etc. [1].

The pseudo Drazin inverse was defined in associative rings [15] as an intermediate between Drazin inverse and generalized Drazin inverse. An element $a \in \mathcal{R}$ is pseudo Drazin invertible if there exists $x \in \mathcal{R}$ such that

$$x \in \text{comm}^2(a), \quad xax = x, \quad \text{and} \quad a^k - a^{k+1}x \in J(\mathcal{R}),$$

for some $k \geq 0$, where $J(\mathcal{R}) = \{b \in \mathcal{R} : 1 + by \in \mathcal{R}^{-1}, \text{ for any } y \in \mathcal{R}\}$ is the Jacobson radical of $\mathcal{R}$. The pseudo Drazin inverse of a is unique, if it exists, and is denoted by a^{pD}. The set of all pseudo Drazin invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{pD}$. Also, $a^\pi = 1 - aa^{pD}$. Recall that, by [9, Corollary 4.2], if $a \in J(\mathcal{R})$ and $b \in \mathcal{R}$, then $ab, ba \in J(\mathcal{R})$. For some interesting properties of the pseudo Drazin inverse see [21, 23].

In 1965, Cline [3] showed that if ab is Drazin invertible, then ba is Drazin invertible too, and the so-called Cline's formula $(ba)^D = b((ab)^D)^2a$ holds. In [11, 13, 19], Cline's formula was generalized to the case of generalized Drazin invertibility.

It is well known that Jacobson's lemma states the following:

Lemma 1.2. *Let $a, b \in \mathcal{R}$. If $1 - ab \in \mathcal{R}^{-1}$, then $1 - ba \in \mathcal{R}^{-1}$ and $(1-ba)^{-1} = 1 + b(1-ab)^{-1}a$.*

Jacobson's lemma has suitable analogues for the group, Drazin, and generalized Drazin inverses [2, 22].

In the case that $aba = aca$, Corach, Duggal, and Harte [4] generalized Jacobson's lemma and proved that if $1 - ac$ is invertible, then $1 - ba$ is invertible too and $(1-ba)^{-1} = 1 + b(1-ac)^{-1}a$. Cline's formula for Drazin and generalized Drazin inverses in a ring under the condition $aba = aca$ was extended in [10, 20].

Yan and Fang [16] investigated invertibility of $1 - ac$ and $1 - bd$, for bounded linear operators between Banach spaces, whenever $acd = dbd$ and $dba = aca$. We observe that, for $d = a$, $aba = aca$. In [14, 17], the generalizations of Jacobson's lemma were given in a ring when $acd = dbd$ and ($bdb = bac$ or $dba = aca$). When $acd = dbd$ and $dba = aca$, Cline's formula for generalized and Drazin invertibility was studied in [12, 19].

In [18], under new conditions $bac = bdb$ and $cdb = cac$ for bounded linear operators between Banach spaces, the authors proved that $1 - ac$ is invertible if and only if $1 - bd$ is invertible. Also, they showed that the previous equivalence holds for the generalized Drazin and Drazin invertibility by spectral properties.

Using conditions introduced in [18], we obtain that $1-bd$ is generalized Drazin invertible if and only if $1-ac$ is generalized Drazin invertible for elements of a ring. Thus, we extend some results from [18] to rings, giving expressions for $(1-bd)^d$ and $(1-ac)^d$, but not using spectral properties. We also present the generalization of Jacobson's lemma for the Drazin inverse, group inverse, and pseudo Drazin inverse in a ring in the case that $bac = bdb$ and $cdb = cac$. The corresponding results are proved for Cline's formula.

2. Jacobson's lemma for Drazin inverses

We study a generalization of Jacobson's lemma for the generalized Drazin inverse in a ring under the assumptions $bac = bdb$ and $cdb = cac$.

Theorem 2.1. *Let $a, b, c, d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$. Then*

$$1 - bd \in \mathcal{R}^d \quad \textit{if and only if} \quad 1 - ac \in \mathcal{R}^d.$$

In this case,

$$(1-ac)^d = (1 - acd(1-bd)^\pi[1-(1-(bd)^2)(1-bd)^\pi]^{-1}b)(1+ac) + acd(1-bd)^d b,$$
$$(1-bd)^d = (1 - bac(1-ac)^\pi[1-(1-(ac)^2)(1-ac)^\pi]^{-1}d)(1+bd) + bac(1-ac)^d d.$$

Proof. Suppose that $\alpha = 1 - bd \in \mathcal{R}^d$ and $\beta = 1 - ac$. By Lemma 1.1, $\alpha\alpha^\pi \in \mathcal{R}^{\text{qnil}}$ and so $1 - (1-(bd)^2)(1-bd)^\pi = 1 - (1+bd)\alpha\alpha^\pi \in \mathcal{R}^{-1}$. Set

$$y = (1 - acd\alpha^\pi[1 - \alpha\alpha^\pi(1+bd)]^{-1}b)(1+ac) + acd\alpha^d b.$$

Then

$$\begin{aligned} y\beta &= 1 - (ac)^2 - acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b(1+ac)\beta + acd\alpha^d b\beta \\ &= 1 - acdb - acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}(1+bd)b\beta + acd\alpha^d\alpha b \\ &= 1 - acd\alpha^\pi b - acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}\alpha^\pi\alpha(1+bd)b \\ &= 1 - acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}(1-\alpha\alpha^\pi(1+bd)+\alpha^\pi\alpha(1+bd))b \\ &= 1 - acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b. \end{aligned}$$

Because bd commutes with α, we have that bd commutes with α^d, α^π, and $[1-\alpha\alpha^\pi(1+bd)]^{-1}$. Now, from

$$\begin{aligned} acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}bacd\alpha^d b &= acd[1-\alpha\alpha^\pi(1+bd)]^{-1}\alpha^\pi bdbd\alpha^d b \\ &= acd[1-\alpha\alpha^\pi(1+bd)]^{-1}\alpha^\pi\alpha^d bdbdb \\ &= 0, \end{aligned}$$

we get

$$\begin{aligned}
y\beta y &= y - acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}by \\
&= y - acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b(1+ac) \\
&\quad + acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}bacd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b(1+ac) \\
&\quad - acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}bacd\alpha^d b \\
&= y - acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}(1+bd)b \\
&\quad + acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}bdbd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}(1+bd)b \\
&= y - acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-2}(\alpha^\pi - \alpha^\pi\alpha(1+bd) - \alpha^\pi(bd)^2)(1+bd)b \\
&= y - acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-2}(\alpha^\pi - \alpha^\pi(1-(bd)^2) - \alpha^\pi(bd)^2)(1+bd)b \\
&= y.
\end{aligned}$$

In order to verify that

$$\beta(1-y\beta) = \beta acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b \in \mathcal{R}^{\text{qnil}},$$

let $z \in \mathcal{R}$ satisfy $\beta(1-y\beta)z = z\beta(1-y\beta)$, i.e. $\beta acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}bz = z\beta acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b$. Since $\alpha^\pi = \alpha^\pi(bd)^2[1-\alpha\alpha^\pi(1+bd)]^{-1}$, we have

$$\begin{aligned}
\alpha\alpha^\pi bzacd &= (bd)^2\alpha\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}bzacd \\
&= bacd\alpha\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}bzacd \\
&= b(\beta acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}bz)acd \\
&= bz\beta acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}bacd \\
&= bzacd\alpha(\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}bdbd) \\
&= bzacd\alpha\alpha^\pi.
\end{aligned}$$

From

$$\begin{aligned}
[1-\alpha\alpha^\pi(1+bd)]^{-1}bzacd\alpha\alpha^\pi &= [1-\alpha\alpha^\pi(1+bd)]^{-1}\alpha\alpha^\pi bzacd \\
&= \alpha\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}bzacd
\end{aligned}$$

and $\alpha\alpha^\pi \in \mathcal{R}^{\text{qnil}}$, we deduce that $1+[1-\alpha\alpha^\pi(1+bd)]^{-1}bz\beta acd\alpha^\pi = 1+[1-\alpha\alpha^\pi(1+bd)]^{-1}bzacd\alpha\alpha^\pi \in \mathcal{R}^{-1}$. Applying Lemma 1.2, we conclude that $1+\beta acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}bz \in \mathcal{R}^{-1}$ and so $\beta acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b \in \mathcal{R}^{\text{qnil}}$.

To prove that $y \in \text{comm}^2(\beta)$, assume that $u\beta = \beta u$, for $u \in \mathcal{R}$. From

$$buacd\alpha = bu\beta acd = b\beta uacd = \alpha buacd,$$

we have that $buacd$ commutes with α^d, α^π, and $[1-\alpha\alpha^\pi(1+bd)]^{-1}$. Using $\alpha^\pi = \alpha^\pi(bd)^2[1-\alpha\alpha^\pi(1+bd)]^{-1}$, we get

$$\begin{aligned} uacd\alpha^\pi b &= uacd(bd)^2\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b \\ &= u(ac)^3d\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b \\ &= (ac)^2uacd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b \\ &= acd(buacd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1})b \\ &= acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}buacdb \\ &= acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}buacac \\ &= acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b(ac)^2u \\ &= acd(\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}(bd)^2)bu \\ &= acd\alpha^\pi bu. \end{aligned}$$

Therefore,

$$uacd\alpha\alpha^d b = acd\alpha\alpha^d bu$$

and

$$uacdbd\alpha\alpha^d b = ac(uacd\alpha\alpha^d b) = acacd\alpha\alpha^d bu = acdbd\alpha\alpha^d bu,$$

which gives $uacd(1+bd)\alpha\alpha^d b = acd(1+bd)\alpha\alpha^d bu$, that is,

$$uacd(1-(bd)^2)\alpha^d b = acd(1-(bd)^2)\alpha^d bu.$$

This equality and

$$\begin{aligned} uacd(bd)^2\alpha^d b = (ac)^2uacd\alpha^d b &= acd(buacd\alpha^d)b = acd\alpha^d buacdb \\ &= acd\alpha^d b(ac)^2u = acd(bd)^2\alpha^d bu \end{aligned}$$

imply

$$uacd\alpha^d b = acd\alpha^d bu. \tag{2.1}$$

Set $v = acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b(1+ac)$. Now, by

$$\begin{aligned} uv &= uacd(bd)^2\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-2}b(1+ac) \\ &= acacuacd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-2}b(1+ac) \\ &= acd(buacd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-2})b(1+ac) \\ &= acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-2}buacdb(1+ac) \\ &= acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-2}bacac(1+ac)u \\ &= acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-2}(bd)^2b(1+ac)u \\ &= acd\alpha^\pi[1-\alpha\alpha^\pi(1+bd)]^{-1}b(1+ac)u \\ &= vu \end{aligned}$$

and (2.1), we have that $uy = yu$ and $y \in \mathrm{comm}^2(\beta)$. Therefore, $\beta \in \mathcal{R}^d$ and $\beta^d = y$.

Similarly, we check that if $\beta \in \mathcal{R}^d$, then $\alpha \in \mathcal{R}^d$ and the equality (2.1) holds. $\square$

Applying Theorem 2.1, we prove that $1-bd$ is Drazin invertible if and only if $1-ac$ is Drazin invertible. When the lower limit of a sum is greater than its upper limit, we define the sum to be 0 (for example, $\sum_{k=0}^{-1} * = 0$) and thus the next result recovers the cases for group inverse and ordinary inverse.

Corollary 2.2. *Let $a,b,c,d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$. Then*

$$1-bd \in \mathcal{R}^D \quad \text{if and only if} \quad 1-ac \in \mathcal{R}^D.$$

In this case, $\operatorname{ind}(1-bd) = \operatorname{ind}(1-ac) = n$,

$$(1-ac)^D = (1-acd(1-bd)^\pi rb)(1+ac) + acd(1-bd)^D b,$$
$$(1-bd)^D = (1-bac(1-ac)^\pi r_1 d)(1+bd) + bac(1-ac)^D d,$$

where $r = \sum_{k=0}^{n-1}[1-(bd)^2]^k$ *and* $r_1 = \sum_{k=0}^{n-1}[1-(ac)^2]^k$.

Proof. By Theorem 2.1, if $\alpha = 1-bd \in \mathcal{R}^D$ and $\operatorname{ind}(1-bd) = n$, then $\beta = 1-ac \in \mathcal{R}^d$ and $(1-ac)^d = y$, where $y = (1-acd\alpha^\pi[1-\alpha(1+bd)\alpha^\pi]^{-1}b)(1+ac) + acd\alpha^D b$. We can easily show that

$$[1-\alpha^\pi\alpha(1+bd)]^{-1} = \sum_{k=0}^{n-1}[\alpha^\pi\alpha(1+bd)]^k,$$

which yields $y = (1-acd\alpha^\pi rb)(1+ac) + acd\alpha^D b$. Further,

$$\beta^{n+1}y = \beta^n(1-acd\alpha^\pi rb) = \beta^n - acd\alpha^n\alpha^\pi rb = \beta^n$$

implies that $y = (1-ac)^D$ and $\operatorname{ind}(\beta) \leq \operatorname{ind}(\alpha)$.

In a similar manner, we verify that $\beta \in \mathcal{R}^D$ gives $\alpha \in \mathcal{R}^D$ and $\operatorname{ind}(\beta) \geq \operatorname{ind}(\alpha)$. □

For $n = 1$ or $n = 0$ in Corollary 2.2, we get generalizations of Jacobson's lemma for the group inverse and ordinary inverse when $bac = bdb$ and $cdb = cac$.

Corollary 2.3. *Let $a,b,c,d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$. Then:*

(i)
$$1-bd \in \mathcal{R}^\# \quad \text{if and only if} \quad 1-ac \in \mathcal{R}^\#.$$

In this case,

$$(1-ac)^\# = (1-acd(1-bd)^\pi b)(1+ac) + acd(1-bd)^\# b,$$
$$(1-bd)^\# = (1-bac(1-ac)^\pi d)(1+bd) + bac(1-ac)^\# d.$$

(ii)
$$1-bd \in \mathcal{R}^{-1} \quad \text{if and only if} \quad 1-ac \in \mathcal{R}^{-1}.$$

In this case,

$$(1-ac)^{-1} = 1 + ac + acd(1-bd)^{-1}b,$$
$$(1-bd)^{-1} = 1 + bd + bac(1-ac)^{-1}d.$$

Under the conditions $bac = bdb$ and $cdb = cac$, we now study Jacobson's lemma for the pseudo Drazin inverse.

Theorem 2.4. *Let $a, b, c, d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$. Then*

$$1 - bd \in \mathcal{R}^{pD} \quad \textit{if and only if} \quad 1 - ac \in \mathcal{R}^{pD}.$$

In this case,

$$\begin{aligned}(1-ac)^{pD} &= (1 - acd(1-bd)^{\pi}[1-(1-(bd)^2)(1-bd)^{\pi}]^{-1}b)(1+ac) \\ &\quad + acd(1-bd)^{pD}b, \\ (1-bd)^{pD} &= (1 - bac(1-ac)^{\pi}[1-(1-(ac)^2)(1-ac)^{\pi}]^{-1}d)(1+bd) \\ &\quad + bac(1-ac)^{pD}d.\end{aligned} \tag{2.2}$$

Proof. Let $\alpha = 1 - bd \in \mathcal{R}^{pD}$, $\beta = 1 - ac$, and y be the right hand side of (2.2). As in the proof of Theorem 2.1, we show that $y\beta y = y$ and $y \in \mathrm{comm}^2(\beta)$. Since $\alpha^k \alpha^{\pi} \in J(\mathcal{R})$ for some $k \geq 0$, we obtain

$$\beta^k\beta^{\pi} = \beta^k acd\alpha^{\pi}[1-\alpha\alpha^{\pi}(1+bd)]^{-1}b = acd\alpha^k\alpha^{\pi}[1-\alpha\alpha^{\pi}(1+bd)]^{-1}b \in J(\mathcal{R}),$$

which gives that $\beta \in \mathcal{R}^{pD}$ and $\beta^{pD} = y$.

The converse implication can be proved similarly. □

3. Cline's formula for Drazin inverses

To present an extension of Cline's formula for the generalized Drazin inverse in the case that $bac = bdb$ and $cdb = cac$, we first prove an auxiliary result related to quasinilpotent elements.

Lemma 3.1. *Let $a, b, c, d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$. Then*

$$bd \in \mathcal{R}^{\mathrm{qnil}} \quad \textit{if and only if} \quad ac \in \mathcal{R}^{\mathrm{qnil}}.$$

Proof. Assume that $ac \in \mathcal{R}^{\mathrm{qnil}}$ and, for $z \in \mathcal{R}$, $zbd = bdz$. Then

$$(acdz^3b)ac = acdz^3bdb = acdbdz^3b = ac(acdz^3b)$$

implies that $1+(acdz^3b)ac \in \mathcal{R}^{-1}$. Using Lemma 1.2, we deduce that $1+z^3bacacd \in \mathcal{R}^{-1}$. Because $1 + zbd$ and $1 - zbd + z^2bdbd$ commute and

$$(1 + zbd)(1 - zbd + z^2bdbd) = 1 + z^3(bd)^3 = 1 + z^3bacacd \in \mathcal{R}^{-1},$$

we have that $1 + zbd \in \mathcal{R}^{-1}$ and so $bd \in \mathcal{R}^{\mathrm{qnil}}$.

If $bd \in \mathcal{R}^{\mathrm{qnil}}$, by [10, Lemma 2.2] (or [19, Lemma 2.6]), $db \in \mathcal{R}^{\mathrm{qnil}}$. By the first part of this proof, we get $ca \in \mathcal{R}^{\mathrm{qnil}}$. Applying again [10, Lemma 2.2], $ac \in \mathcal{R}^{\mathrm{qnil}}$. □

Now, we can give a new generalization of Cline's formula for the generalized Drazin inverse in a ring.

Theorem 3.2. *Let $a, b, c, d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$. Then*

$$bd \in \mathcal{R}^d \quad \text{if and only if} \quad ac \in \mathcal{R}^d.$$

In this case,

$$(ac)^d = acd[(bd)^d]^3 b \qquad \text{and} \qquad (bd)^d = b[(ac)^d]^2 d.$$

Proof. Let $bd \in \mathcal{R}^d$ and $y = acd[(bd)^d]^3 b$. Firstly, notice that

$$yacy = acd[(bd)^d]^3 bacacd[(bd)^d]^3 b = acd[(bd)^d]^3 (bd)^3 [(bd)^d]^3 b = y.$$

To check that $y \in \operatorname{comm}^2(ac)$, suppose that $zac = acz$, for $z \in \mathcal{R}$. Since

$$(bzacd)bd = bzacacd = baczacd = bd(bzacd),$$

$bzacd$ commutes with $(bd)^d$ and thus

$$\begin{aligned} yz &= acd[(bd)^d]^3 bz = acd[(bd)^d]^5 bdbdbz = acd[(bd)^d]^5 bzacac \\ &= acd([(bd)^d]^5 bzacd)b = acdbzacd[(bd)^d]^5 b = zacacacd[(bd)^d]^5 b \\ &= zacdbdbd[(bd)^d]^5 b = zacd[(bd)^d]^3 b = zy. \end{aligned}$$

In order to verify that

$$ac(1 - acy) = ac(1 - dbdbd[(bd)^d]^3 b) = ac(1 - d(bd)^d b) \in \mathcal{R}^{\text{qnil}},$$

set $c' = c(1 - d(bd)^d b)$ and $d' = d(1 - bd(bd)^d)$. Now, we get

$$bac' = bac(1 - d(bd)^d b) = bd(1 - bd(bd)^d)b = bd'b$$

and

$$\begin{aligned} c'ac' &= c'ac(1 - d(bd)^d b) = (cac - cd(bd)^d bac)(1 - d(bd)^d b) \\ &= c(1 - d(bd)^d b)db(1 - d(bd)^d b) = c'd(1 - bd(bd)^d)b = c'd'b. \end{aligned}$$

From $bd' = bd(1 - bd(bd)^d) \in \mathcal{R}^{\text{qnil}}$ and Lemma 3.1, we conclude that $ac(1 - d(bd)^d b) = ac' \in \mathcal{R}^{\text{qnil}}$. Hence, $ac \in \mathcal{R}^d$ and $(ac)^d = y$.

If $ac \in \mathcal{R}^d$, then, by [10, Theorem 2.3] (or [19, Theorem 2.7]), $ca \in \mathcal{R}^d$. By the previous part of this proof, we have that $db \in \mathcal{R}^d$ and

$$(db)^d = dba[(ca)^d]^3 c = db[(ac)^d]^3 ac = db[(ac)^d]^2.$$

Using again [10, Theorem 2.3], $bd \in \mathcal{R}^d$ and

$$\begin{aligned} (bd)^d &= b[(db)^d]^2 d = bdb[(ac)^d]^2 db[(ac)^d]^2 d = bac[(ac)^d]^3 acdb[(ac)^d]^2 d \\ &= b[(ac)^d]^2 acac[(ac)^d]^2 d = b[(ac)^d]^2 d. \end{aligned}$$

□

Using Theorem 3.2, we obtain Cline's formula for the Drazin inverse under the assumptions $bac = bdb$ and $cdb = cac$.

Corollary 3.3. *Let $a, b, c, d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$. Then*

$$bd \in \mathcal{R}^D \quad \text{if and only if} \quad ac \in \mathcal{R}^D.$$

In this case, $(ac)^D = acd[(bd)^D]^3 b$, $(bd)^D = b[(ac)^D]^2 d$, and

$$\operatorname{ind}(ac) - 2 \le \operatorname{ind}(bd) \le \operatorname{ind}(ac) + 1.$$

Proof. Applying Theorem 3.2, $bd \in \mathcal{R}^D$ implies $ac \in \mathcal{R}^d$ and $(ac)^d = acd[(bd)^D]^3b$. For $n = \operatorname{ind}(bd)$,

$$(ac)^{n+2}(1 - ac(ac)^d) = acd(bd)^n(1 - bd(bd)^D)b = 0,$$

which gives $ac \in \mathcal{R}^D$ and $\operatorname{ind}(ac) \leq \operatorname{ind}(bd) + 2$.

Similarly, we prove that if $ac \in \mathcal{R}^D$, then $bd \in \mathcal{R}^D$ and $\operatorname{ind}(bd) \leq \operatorname{ind}(ac)+1$. □

In the following theorem, we consider a new extension of Cline's formula for the pseudo Drazin inverse.

Theorem 3.4. *Let $a, b, c, d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$. Then*

$$bd \in \mathcal{R}^{pD} \quad \text{if and only if} \quad ac \in \mathcal{R}^{pD}.$$

In this case,

$$(ac)^{pD} = acd[(bd)^{pD}]^3b \qquad \text{and} \qquad (bd)^{pD} = b[(ac)^{pD}]^2d.$$

Proof. Assume that $bd \in \mathcal{R}^{pD}$. For $y = acd[(bd)^{pD}]^3b$, we get $yacy = y$ and $y \in \operatorname{comm}^2(ac)$ as in the proof of Theorem 3.2. Further, because $(bd)^k(1-bd(bd)^{pD}) \in J(\mathcal{R})$ for some $k \geq 0$, we have that

$$(ac)^{k+2}(1 - acy) = acd(bd)^k(1 - bd(bd)^{pD})b \in J(\mathcal{R}).$$

Therefore, $ac \in \mathcal{R}^{pD}$ and $(ac)^{pD} = y$.

In the same way, we show that $ac \in \mathcal{R}^{pD}$ implies $bd \in \mathcal{R}^{pD}$. □

We remark that, for $b = c$ in the previous results, we recover some results from [4, 10, 20], and, for $b = c$ and $a = d$, results from [2, 3, 11, 13, 15, 19, 22]. Also, in these cases, we can get expressions for the corresponding inverses.

References

[1] S. L. Campbell and C. D. Meyer, Jr., *Generalized inverses of linear transformations*, corrected reprint of the 1979 original, Dover, New York, 1991. MR 1105324.

[2] N. Castro-González, C. Mendes-Araújo and P. Patricio, Generalized inverses of a sum in rings, *Bull. Aust. Math. Soc.* **82** (2010), no. 1, 156–164. MR 2669152.

[3] R. E. Cline, *An application of representations for the generalized inverse of a matrix*, MRC Technical Summary Report 592, Mathematics Research Center, Univ. of Wisconsin–Madison, 1965.

[4] G. Corach, B. Duggal and R. Harte, Extensions of Jacobson's lemma, *Comm. Algebra* **41** (2013), no. 2, 520–531. MR 3011779.

[5] M. P. Drazin, Pseudo-inverses in associative rings and semigroups, *Amer. Math. Monthly* **65** (1958), 506–514. MR 0098762.

[6] R. Harte, On quasinilpotents in rings, *Panamer. Math. J.* **1** (1991), 10–16. MR 1088863.

[7] J. J. Koliha, A generalized Drazin inverse, *Glasgow Math. J.* **38** (1996), no. 3, 367–381. MR 1417366.

[8] J. J. Koliha and P. Patricio, Elements of rings with equal spectral idempotents, *J. Aust. Math. Soc.* **72** (2002), no. 1, 137–152. MR 1868712.

[9] T. Y. Lam, *A first course in noncommutative rings*, second edition, Graduate Texts in Mathematics, 131, Springer-Verlag, New York, 2001. MR 1838439.

[10] H. Lian and Q. Zeng, An extension of Cline's formula for a generalized Drazin inverse, *Turkish J. Math.* **40** (2016), no. 1, 161–165. MR 3438793.

[11] Y. Liao, J. Chen and J. Cui, Cline's formula for the generalized Drazin inverse, *Bull. Malays. Math. Sci. Soc. (2)* **37** (2014), no. 1, 37–42. MR 3151689.

[12] V. G. Miller and H. Zguitti, New extensions of Jacobson's lemma and Cline's formula, *Rend. Circ. Mat. Palermo (2)* **67** (2018), no. 1, 105–114. MR 3777821.

[13] D. Mosić, A note on Cline's formula for the generalized Drazin inverse, *Linear Multilinear Algebra* **63** (2015), no. 6, 1106–1110. MR 3291958.

[14] D. Mosić, Extensions of Jacobson's lemma for Drazin inverses, *Aequationes Math.* **91** (2017), no. 3, 419–428. MR 3651555.

[15] Z. Wang and J. Chen, Pseudo Drazin inverses in associative rings and Banach algebras, *Linear Algebra Appl.* **437** (2012), no. 6, 1332–1345. MR 2942354.

[16] K. Yan and X. C. Fang, Common properties of the operator products in local spectral theory, *Acta Math. Sin. (Engl. Ser.)* **31** (2015), no. 11, 1715–1724. MR 3406670.

[17] K. Yan, Q. Zeng and Y. Zhu, Generalized Jacobson's lemma for Drazin inverses and its applications, *Linear Multilinear Algebra* **68** (2020), no. 1, 81–93. MR 4037073.

[18] K. Yan, Q. Zeng and Y. Zhu, On Drazin spectral equation for the operator products, *Complex Anal. Oper. Theory* **14** (2020), no. 1, Paper No. 12, 15 pp. MR 4056941.

[19] Q. Zeng, Z. Wu and Y. Wen, New extensions of Cline's formula for generalized inverses, *Filomat* **31** (2017), no. 7, 1973–1980. MR 3635233.

[20] Q. Zeng and H. Zhong, New results on common properties of the products AC and BA, *J. Math. Anal. Appl.* **427** (2015), no. 2, 830–840. MR 3323010.

[21] H. Zhu, J. Chen and P. Patrício, Representations for the pseudo Drazin inverse of elements in a Banach algebra, *Taiwanese J. Math.* **19** (2015), no. 2, 349–362. MR 3332301.

[22] G. Zhuang, J. Chen and J. Cui, Jacobson's lemma for the generalized Drazin inverse, *Linear Algebra Appl.* **436** (2012), no. 3, 742–746. MR 2854904.

[23] H. Zou and J. Chen, On the pseudo Drazin inverse of the sum of two elements in a Banach algebra, *Filomat* **31** (2017), no. 7, 2011–2022. MR 3635237.

Dijana Mosić
Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia
dijana@pmf.ni.ac.rs

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