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$\mathfrak{D}^\perp$-INVARIANT REAL HYPERSURFACES IN COMPLEX GRASSMANNIANS OF RANK TWO

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ABSTRACT. Let M be a real hypersurface in a complex Grassmannian of rank two. Denote by $\mathfrak{J}$ the quaternionic Kähler structure of the ambient space, $TM^\perp$ the normal bundle over M, and $\mathfrak{D}^\perp = \mathfrak{J}TM^\perp$. The real hypersurface M is said to be $\mathfrak{D}^\perp$-invariant if $\mathfrak{D}^\perp$ is invariant under the shape operator of M. We show that if M is $\mathfrak{D}^\perp$-invariant, then M is Hopf. This improves the results of Berndt and Suh [*Int. J. Math.* **23** (2012) 1250103] and [*Monatsh. Math.* **127** (1999), 1–14]. We also classify $\mathfrak{D}^\perp$ real hypersurfaces in complex Grassmannians of rank two of noncompact type with constant principal curvatures.

1. Introduction

Denote by $\hat{M}^m(c)$ the compact complex Grassmannian $SU_{m+2}/S(U_2U_m)$ of rank two (resp. noncompact complex Grassmannian $SU_{2,m}/S(U_2U_m)$ of rank two) for $c > 0$ (resp. $c < 0$), where $c = \max|K|/8$ is a scaling factor for the Riemannian metric g and K is the sectional curvature for $\hat{M}^m(c)$. It is well known that $\hat{M}^m(c)$ is a Riemannian symmetric space equipped with a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$. A tangent vector $X \in T_x\hat{M}^m(c)$, $x \in \hat{M}^m(c)$, is said to be *singular* if either $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

Let M be a connected real hypersurface in $\hat{M}^m(c)$. Then the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$ naturally induce two subbundles $JTM^\perp$ and $\mathfrak{J}TM^\perp$ in the tangent bundle TM over M. Denote by A the shape operator on M. In [3] and [4], Berndt and Suh studied real hypersurfaces M in $\hat{M}^m(c)$ under the conditions:

(I) $A\mathfrak{J}TM^\perp \subset \mathfrak{J}TM^\perp$;
(II) $AJTM^\perp \subset JTM^\perp$.

Recall that a Hopf hypersurface is a real hypersurface which satisfies the condition (II). Let $\mathfrak{D}^\perp = \mathfrak{J}TM^\perp$. A real hypersurface M is said to be $\mathfrak{D}^\perp$-invariant if it satisfies the condition (I). The following theorems provide a list of possible real hypersurfaces satisfying these two conditions.

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Theorem 1.1 ([4]). *Let M be a connected real hypersurface in $SU_{m+2}/S(U_2U_m)$, $m \geq 3$. Then M is Hopf and $\mathfrak{D}^\perp$-invariant if and only if one of the following holds:*

(A) M is an open part of a tube around a totally geodesic $SU_{m+1}/S(U_2U_{m-1})$ in $SU_{m+2}/S(U_2U_m)$, or

(B) M is an open part of a tube around a totally geodesic $\mathbb{H}P^n = Sp_{n+1}/Sp_1Sp_n$ in $SU_{m+2}/S(U_2U_m)$, where $m = 2n$ is even.

Theorem 1.2 ([3]). *Let M be a connected real hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 2$. Then M is Hopf and $\mathfrak{D}^\perp$-invariant if and only if one of the following holds:*

(A) M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$, or

(B) M is an open part of a tube around a totally geodesic $\mathbb{H}H^n = Sp_{1,n}/Sp_1Sp_n$ in $SU_{2,m}/S(U_2U_m)$, where $m = 2n$ is even, or

(C_1) M is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$, or

(C_2) M is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JX \perp \mathfrak{J}X$, or

(D) the normal vector N of M at each point $x \in M$ is singular of type $JN \perp \mathfrak{J}N$. Moreover, M has at least four distinct principal curvatures, three of which are given by

$$\alpha = 2\sqrt{-c}, \quad \gamma = 0, \quad \beta = \sqrt{-c},$$

with corresponding principal curvature spaces

$$T_\alpha = JTM^\perp \oplus \mathfrak{J}TM^\perp, \quad T_\gamma = \mathfrak{J}JTM^\perp, \quad \text{and} \quad T_\beta \perp JTM^\perp \oplus \mathfrak{J}TM^\perp \oplus \mathfrak{J}JTM^\perp.$$

If μ is another (possibly nonconstant) principal curvature function, then $JT_\mu \subset T_\beta$ and $\mathfrak{J}T_\mu \subset T_\beta$.

Real hypersurfaces of type (A), (B), (C_1), and (C_2) in Theorem 1.2 and Theorem 1.1 have been the main focus along this line in the past two decades. Finding the simplest conditions characterizing real hypersurfaces in these theorems (or any of its subclasses) has become a key step in the study of such real hypersurfaces.

Observe that the unit normal vector field N for real hypersurfaces M appearing in these theorems is singular. In this line of thought, for a real hypersurface M in $\hat{M}^m(c)$, Lee and Suh showed that if M is Hopf and N is singular of type $JN \perp \mathfrak{J}N$ everywhere, then it is $\mathfrak{D}^\perp$-invariant (cf. [5, 9]). On the other hand, for the case $c > 0$, it was shown in [7] that the condition (II) is necessary for the condition (I) and $JN \in \mathfrak{J}N$ everywhere.

In this paper, we show that the condition (II) is unnecessary in these two theorems, as the following theorem asserts.

Theorem 1.3. *Let M be a connected real hypersurface in $\hat{M}^m(c)$, $m \geq 2$. If M is $\mathfrak{D}^\perp$-invariant, then it satisfies the condition (II), that is, M is Hopf.*

It is worthwhile to remark that there is no known example for Case (D) in Theorem 1.2. In [3], Berndt and Suh conjectured that such a real hypersurface does not exist. With the assumption of the principal curvatures being constant, we prove the nonexistence of such real hypersurfaces and this gives a partial answer to the conjecture. More precisely, we have the following result.

Theorem 1.4. *Let M be a connected real hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 2$. If M is $\mathfrak{D}^\perp$-invariant and has constant principal curvatures, then one of the following holds:*

- *(A) M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$, or*
- *(B) M is an open part of a tube around a totally geodesic $\mathbb{H}H^n = Sp_{1,n}/Sp_1Sp_n$ in $SU_{2,m}/S(U_2U_m)$, where $m = 2n$ is even, or*
- *(C_1) M is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$, or*
- *(C_2) M is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JX \perp \mathfrak{J}X$.*

Remark 1.1. Similar results on real hypersurfaces in quaternionic space forms were obtained in [1].

2. Preliminaries

In this section, we recall some fundamental identities for real hypersurfaces in complex Grassmannians of rank two, which were proved in [3, 4, 6, 8].

Let $\hat{M}^m(c)$ be the compact complex Grassmannian $SU_{m+2}/S(U_2U_m)$ of rank two (resp. noncompact complex Grassmannian $SU_{2,m}/S(U_2U_m)$ of rank two) for $c > 0$ (resp. $c < 0$), where $c = \max|K|/8$ is a scaling factor for the Riemannian metric g and K is the sectional curvature for $\hat{M}^m(c)$. The Riemannian geometry of $\hat{M}^m(c)$ was studied in [2, 3, 4]. Denote by J and $\mathfrak{J}$ the Kähler structure J and quaternionic Kähler structure on $\hat{M}^m(c)$, respectively.

Let M be a connected, oriented real hypersurface isometrically immersed in $\hat{M}^m(c)$, $m \geq 2$, and let N be a unit normal vector field on M. Denote by the same g the Riemannian metric on M. The almost contact metric 3-structure $(\phi_a, \xi_a, \eta_a, g)$ on M is given by

$$J_aX = \phi_aX + \eta_a(X)N, \quad J_aN = -\xi_a, \quad \eta_a(X) = g(X, \xi_a),$$

for any $X \in TM$, where $\{J_1, J_2, J_3\}$ is a canonical local basis of $\mathfrak{J}$. It follows that

$$\begin{aligned} \phi_a\phi_{a+1} - \xi_a \otimes \eta_{a+1} &= \phi_{a+2}, \\ \phi_a\xi_{a+1} = \xi_{a+2} &= -\phi_{a+1}\xi_a, \end{aligned}$$

for $a \in \{1, 2, 3\}$. The indices in the preceding equations are taken modulo three.

The Kähler structure J induces on M an almost contact metric structure by

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \quad \eta(X) = g(X, \xi).$$

Let $\mathfrak{D}$ be the orthogonal complement of $\mathfrak{D}^\perp$ in TM. We define a local $(1,1)$-tensor field θ_a on M by

$$\theta_a := \phi_a\phi - \xi_a \otimes \eta.$$

Denote by ∇ the Levi-Civita connection on M. Then there exist local 1-forms q_a, $a \in \{1,2,3\}$, such that

$$\left.\begin{aligned}\nabla_X \xi &= \phi A X\\ \nabla_X \xi_a &= \phi_a A X + q_{a+2}(X)\xi_{a+1} - q_{a+1}(X)\xi_{a+2}\\ \nabla_X \phi\xi_a &= \theta_a A X + \eta_a(\xi) A X + q_{a+2}(X)\phi\xi_{a+1} - q_{a+1}(X)\phi\xi_{a+2}.\end{aligned}\right\} \tag{2.1}$$

The following identities are known.

Lemma 2.1 ([6])**.**

(a) θ_a *is symmetric,*
(b) $\phi\xi_a = \phi_a\xi$*,*
(c) $\theta_a\xi = -\xi_a$*,* $\theta_a\xi_a = -\xi$*,* $\theta_a\phi\xi_a = \eta(\xi_a)\phi\xi_a$*,*
(d) $\theta_a\xi_{a+1} = \phi\xi_{a+2} = -\theta_{a+1}\xi_a$*,*
(e) $-\theta_a\phi\xi_{a+1} + \eta(\xi_{a+1})\phi\xi_a = \xi_{a+2} = \theta_{a+1}\phi\xi_a - \eta(\xi_a)\phi\xi_{a+1}$*.*

Lemma 2.2 ([6])**.** *If* $\xi \in \mathfrak{D}$ *everywhere, then* $A\phi\xi_a = 0$ *for* $a \in \{1,2,3\}$*.*

For each $x \in M$, we define a subspace $\mathcal{H}^\perp$ of T_xM by

$$\mathcal{H}^\perp := \operatorname{span}\{\xi, \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3\}.$$

Let $\mathcal{H}$ be the orthogonal complement of $\mathcal{H}^\perp$ in T_xM. Then $\dim\mathcal{H} = 4m-4$ (resp. $\dim\mathcal{H} = 4m-8$) when $\xi \in \mathfrak{D}^\perp$ (resp. $\xi \notin \mathfrak{D}^\perp$). Moreover, $\theta_{a|\mathcal{H}}$ has two eigenvalues: 1 and -1. Denote by $\mathcal{H}_a(\varepsilon)$ the eigenspace corresponding to the eigenvalue ε of $\theta_{a|\mathcal{H}}$. Then $\dim\mathcal{H}_a(1) = \dim\mathcal{H}_a(-1)$ is even, and

$$\begin{aligned}&\phi\mathcal{H}_a(\varepsilon) = \phi_a\mathcal{H}_a(\varepsilon) = \theta_a\mathcal{H}_a(\varepsilon) = \mathcal{H}_a(\varepsilon),\\ &\phi_b\mathcal{H}_a(\varepsilon) = \theta_b\mathcal{H}_a(\varepsilon) = \mathcal{H}_a(-\varepsilon), \quad (a \neq b).\end{aligned}$$

The equations of Gauss and Codazzi are respectively given by

$$\begin{aligned}R(X,Y)Z = {}& g(AY,Z)AX - g(AX,Z)AY + c\{g(Y,Z)X - g(X,Z)Y\\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\}\\ &+ c\sum_{a=1}^{3}\big\{g(\phi_a Y,Z)\phi_a X - g(\phi_a X,Z)\phi_a Y - 2g(\phi_a X,Y)\phi_a Z\\ &+ g(\theta_a Y,Z)\theta_a X - g(\theta_a X,Z)\theta_a Y\big\},\end{aligned}$$

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = {}& c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\}\\ &+ c\sum_{a=1}^{3}\big\{\eta_a(X)\phi_a Y - \eta_a(Y)\phi_a X - 2g(\phi_a X,Y)\xi_a\\ &+ \eta_a(\phi X)\theta_a Y - \eta_a(\phi Y)\theta_a X\big\}.\end{aligned}$$

3. Proof of Theorem 1.3

Under the assumption that $A\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$, after having a suitable choice of canonical local basis $\{J_1, J_2, J_3\}$ for $\mathfrak{J}$, the (local) vector fields ξ_1, ξ_2, ξ_3 are principal, say $A\xi_a = \beta_a \xi_a$, for $a \in \{1,2,3\}$. By using the Codazzi equation of such real hypersurfaces, we have (cf. [3, 4]):

$$\begin{aligned}
&\beta_a g((\phi_a A + A\phi_a)X, Y) - 2g(A\phi_a AX, Y) \\
&\qquad - (\beta_a - \beta_{a+1}) q_{a+2}(\xi_a)\{\eta_a(X)\eta_{a+1}(Y) - \eta_{a+1}(X)\eta_a(Y)\} \\
&\qquad - (\beta_a - \beta_{a+2}) q_{a+1}(\xi_a)\{\eta_{a+2}(X)\eta_a(Y) - \eta_a(X)\eta_{a+2}(Y)\} \\
&\quad = c\big(-2\eta(\xi_a)g(\phi X, Y) - 2g(\phi_a X, Y) + 2\eta(X)\eta_a(\phi Y) - 2\eta(Y)\eta_a(\phi X) \\
&\qquad + 2\eta_{a+1}(X)\eta_{a+2}(Y) - 2\eta_{a+1}(Y)\eta_{a+2}(X) \\
&\qquad + 2\eta_{a+1}(\phi X)\eta_{a+2}(\phi Y) - 2\eta_{a+1}(\phi Y)\eta_{a+2}(\phi X) \\
&\qquad + 2\eta_a(Y)\,\{2\eta(\xi_a)\eta_a(\phi X) - \eta(\xi_{a+1})\eta_{a+1}(\phi X) - \eta(\xi_{a+2})\eta_{a+2}(\phi X)\} \\
&\qquad - 2\eta_a(X)\,\{2\eta(\xi_a)\eta_a(\phi Y) - \eta(\xi_{a+1})\eta_{a+1}(\phi Y) - \eta(\xi_{a+2})\eta_{a+2}(\phi Y)\} \\
&\qquad - (\beta_a - \beta_{a+1})\{q_{a+2}(X)\eta_{a+1}(Y) - q_{a+2}(Y)\eta_{a+1}(X)\} \\
&\qquad + (\beta_a - \beta_{a+2})\{q_{a+1}(X)\eta_{a+2}(Y) - q_{a+1}(Y)\eta_{a+2}(X)\}\big)
\end{aligned} \tag{3.1}$$

for all X, Y tangent to M.

We consider two cases: (i) $\xi \in \mathfrak{D}^\perp$ everywhere, and (ii) $\xi \notin \mathfrak{D}^\perp$ at some points of M.

Case (i): Suppose $\xi \in \mathfrak{D}^\perp$ everywhere. For each $x \in M$, since $\mathfrak{D}$ is invariant under A, ϕ, and ϕ_a, from (3.1), we have

$$\begin{aligned}
&2c\{\eta(\xi_a)\phi X + \phi_a X\} + \beta_a(\phi_a A + A\phi_a)X - 2A\phi_a AX \\
&\qquad\qquad = -(\beta_a - \beta_{a+1})q_{a+2}(X)\xi_{a+1} + (\beta_a - \beta_{a+2})q_{a+1}(X)\xi_{a+2},
\end{aligned}$$

for all $X \in \mathfrak{D}$ and $a \in \{1,2,3\}$. Since the left-hand side is in $\mathfrak{D}$ and the right-hand side is in $\mathfrak{D}^\perp$, we have

$$2c\{\eta(\xi_a)\phi X + \phi_a X\} + \beta_a(\phi_a A + A\phi_a)X - 2A\phi_a AX = 0 \tag{3.2}$$
$$(\beta_a - \beta_{a+1})q_{a+2}(X) = 0,$$

for all $X \in \mathfrak{D}$ and $a \in \{1,2,3\}$. Note that if $X \in \mathfrak{D}$, then $\phi_a X \in \mathfrak{D}$. Next, applying ϕ_a on both sides of (3.2) and replacing X by $\phi_a X$ in (3.2) give

$$2c\{\eta(\xi_a)\theta_a X - X\} - \beta_a AX + \beta_a \phi_a A\phi_a X - 2\phi_a A\phi_a AX = 0 \tag{3.3}$$

and

$$2c\{\eta(\xi_a)\theta_a X - X\} + \beta_a \phi_a A\phi_a X - \beta_a AX - 2A\phi_a A\phi_a X = 0,$$

respectively. Hence,

$$(\phi_a A\phi_a)AX = A(\phi_a A\phi_a)X, \quad a \in \{1,2,3\}, \tag{3.4}$$

for all $X \in \mathfrak{D}$. With the assumption that $\xi \in \mathfrak{D}^\perp$, we have $\mathfrak{D} = \mathcal{H}$ and $A\mathcal{H} \subset \mathcal{H}$. By (3.4), there exist common orthonormal eigenvectors $X_1, \ldots, X_{4m-4} \in \mathcal{H}$ of A

and $\phi_1 A\phi_1$. It follows that $AX_j = \lambda_j X_j$ and $A\phi_1 X_j = \mu_j \phi_1 X_j$. Using these in (3.3), we have

$$2c\eta(\xi_1)\theta_1 X_j - (2c + \lambda_j\beta_1 + \mu_j\beta_1 - 2\lambda_j\mu_j)X_j = 0.$$

Since $\xi \in \mathfrak{D}^\perp$, we suppose $\eta(\xi_1) \neq 0$ without loss of generality. Then

$$\theta_1 X_j - \varepsilon X_j = 0,$$

where $\varepsilon = (2c + \beta_1(\lambda_j + \mu_j) - 2\lambda_j\mu_j)/2c\eta(\xi_1)$. This implies that ε is an eigenvalue of θ_1. Hence $\varepsilon \in \{1, -1\}$. Without loss of generality, we can assume that

$$X_1, \dots, X_{2m-2} \in \mathcal{H}_1(1) \qquad \text{and} \qquad X_{2m-1}, \dots, X_{4m-4} \in \mathcal{H}_1(-1).$$

Consequently, $A\mathcal{H}_1(1) \subset \mathcal{H}_1(1)$ and hence $\phi_2 A\phi_2\mathcal{H}_1(1) \subset \mathcal{H}_1(1)$. Thus, if we take $a = 2$ in (3.4), then there exist orthonormal vectors $\tilde{X}_1, \dots, \tilde{X}_{2m-2} \in \mathcal{H}_1(1)$ such that $A\tilde{X}_j = \tilde{\lambda}_j\tilde{X}_j$ and $A\phi_2\tilde{X}_j = \tilde{\mu}_j\phi_2\tilde{X}_j$. From (3.3), we have

$$2c\eta(\xi_2)\theta_2\tilde{X}_j - (2c + \tilde{\lambda}_j\beta_2 + \tilde{\mu}_j\beta_2 - 2\tilde{\lambda}_j\tilde{\mu}_j)\tilde{X}_j = 0.$$

Since $\tilde{X}_j \in \mathcal{H}_1(1)$ and $\theta_2\tilde{X}_j \in \mathcal{H}_1(-1)$, we have $\eta(\xi_2) = 0$. In a similar manner, we obtain $\eta(\xi_3) = 0$. Thus, we have $\eta(\xi_1) = \pm 1$ or $\xi = \pm\xi_1$. As a result, we have shown that $A\xi = \beta_1\xi$ at each $x \in M$. Hence M is Hopf.

Case (ii): Suppose that $\xi \notin \mathfrak{D}^\perp$ at a point $x \in M$. Since $\mathfrak{D}$ is invariant under A and ϕ_a, after letting $Y \in \mathfrak{D}$ and $X = \xi_{a+1}$ in (3.1), we have

$$(\beta_a - \beta_{a+1})q_{a+2}(Y) = cg\big(2\eta_{a+1}(\xi)\phi\xi_a + 4\eta_a(\xi)\phi\xi_{a+1} + 2\eta_{a+2}(\xi)\xi, Y\big). \tag{3.5}$$

Similarly, if we let $Y \in \mathfrak{D}$ and $X = \xi_{a+2}$ in (3.1), then

$$(\beta_{a+2} - \beta_a)q_{a+1}(Y) = cg\big(4\eta_a(\xi)\phi\xi_{a+2} + 2\eta_{a+2}(\xi)\phi\xi_a - 2\eta_{a+1}(\xi)\xi, Y\big).$$

Raising the index of this equation by one gives

$$(\beta_a - \beta_{a+1})q_{a+2}(Y) = cg\big(4\eta_{a+1}(\xi)\phi\xi_a + 2\eta_a(\xi)\phi\xi_{a+1} - 2\eta_{a+2}(\xi)\xi, Y\big). \tag{3.6}$$

Denote by P the orthogonal projection from T_xM onto $\mathfrak{D}$. Since Y is an arbitrary vector in $\mathfrak{D}$, by (3.5) and (3.6) we obtain

$$P\left(2\eta_{a+2}(\xi)\xi - \eta_{a+1}(\xi)\phi\xi_a + \eta_a(\xi)\phi\xi_{a+1}\right) = 0,$$

which implies that

$$\begin{aligned} &2\eta_{a+2}(\xi)\xi - \eta_{a+1}(\xi)\phi\xi_a + \eta_a(\xi)\phi\xi_{a+1} \\ &\qquad\qquad - \sum_{b=1}^{3} g\big(2\eta_{a+2}(\xi)\xi - \eta_{a+1}(\xi)\phi\xi_a + \eta_a(\xi)\phi\xi_{a+1}, \xi_b\big)\xi_b = 0. \end{aligned}$$

Since ξ, ξ_1, ξ_2, ξ_3, $\phi\xi_1$, $\phi\xi_2$, $\phi\xi_3$ are linearly independent, we obtain that $\eta_a(\xi) = 0$ for $a \in \{1, 2, 3\}$. This means that $\xi \in \mathfrak{D}$ at $x \in M$. Hence, we conclude that $\xi \in \mathfrak{D}$ everywhere by the connectedness of M and the continuity of $\sum_{a=1}^{3}\eta_a(\xi)^2$.

Since $\eta_a(\xi) = 0$ for $a \in \{1, 2, 3\}$, equations (3.1) and (3.5) give

$$\begin{aligned} \beta_a(\phi_a A + A\phi_a)X - 2A\phi_a AX = 2c\{&-\phi_a X - \eta(X)\phi_a\xi - \eta(\phi_a X)\xi \\ &- \eta_{a+1}(\phi X)\phi\xi_{a+2} + \eta_{a+2}(\phi X)\phi\xi_{a+1}\} \end{aligned}$$

for all $X \in \mathfrak{D}$. First acting by ϕ_a on both sides of the preceding equation and next replacing X by $\phi_a X$ in that equation we get

$$\begin{aligned} \beta_a(-A + \phi_a A \phi_a)X - 2\phi_a A \phi_a A X = 2c\{&X + \eta(X)\xi - \eta(\phi_a X)\phi_a \xi \\ &+ \eta(\phi_{a+1}X)\phi_{a+1}\xi + \eta(\phi_{a+2}X)\phi_{a+2}\xi\} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \beta_a(\phi_a A \phi_a - A)X - 2A\phi_a A \phi_a X = 2c\{&X - \eta(\phi_a X)\phi_a \xi + \eta(X)\xi \\ &+ \eta(\phi_{a+2}X)\phi_{a+2}\xi + \eta(\phi_{a+1}X)\phi_{a+1}\xi\} \end{aligned}$$

for all $X \in \mathfrak{D}$. Hence $(\phi_a A \phi_a)AX = A(\phi_a A \phi_a)X$ for $X \in \mathfrak{D}$. Since $A\mathfrak{D} \subset \mathfrak{D}$ and $\phi_a A \phi_a \mathfrak{D} \subset \mathfrak{D}$, there exists an orthonormal basis $\{X_1, \dots, X_{4m-4}\}$ for $\mathfrak{D}$ such that $AX_j = \lambda_j X_j$ and $A\phi_a X_j = \mu_j \phi_a X_j$. Taking an arbitrary $j \in \{1, 2, \dots, 4m-4\}$ and substituting $X = X_j$ in (3.7), we obtain

$$\begin{aligned} \{2\lambda_j\mu_j - \beta_a(\lambda_j + \mu_j)\}X_j = 2c\{&X_j + \eta(X_j)\xi - \eta(\phi_a X_j)\phi_a \xi \\ &+ \eta(\phi_{a+1}X_j)\phi_{a+1}\xi + \eta(\phi_{a+2}X_j)\phi_{a+2}\xi\}. \end{aligned}$$

Let X_j^H be the $\mathcal{H}$-component of X_j. Then the $\mathcal{H}$- and ξ-components of the preceding equations are given respectively by

$$\begin{aligned} 0 &= \{2\lambda_j\mu_j - \beta_a(\lambda_j + \mu_j) - 2c\}X_j^H, \\ 0 &= \{2\lambda_j\mu_j - \beta_a(\lambda_j + \mu_j) - 4c\}\eta(X_j). \end{aligned}$$

For each $j \in \{1, 2, \dots, 4m-4\}$ with $X_j^H \neq 0$, the former of these equations implies that $2\lambda_j\mu_j - \beta_a(\lambda_j + \mu_j) = 2c$. This, together with the latter equation, gives $\eta(X_j) = 0$. This means that $A\xi \perp \mathcal{H}$. Since $A\phi\xi_a = 0$ for $a \in \{1, 2, 3\}$ by Lemma 2.2, we obtain that ξ is principal on M. This completes the proof.

4. Proof of Theorem 1.4

It is clear that we only need to consider the case $\xi \in \mathfrak{D}$ everywhere. Denote the spectrum of $A_{|\mathcal{H}}$ by $\mathrm{Spec}(\mathcal{H})$. For each $\lambda \in \mathrm{Spec}(\mathcal{H})$, denote by T_λ the subbundle of $\mathcal{H}$ foliated by eigenspaces of $A_{|\mathcal{H}}$ corresponding to λ. We will use Cartan's method to prove the result, and begin with some fundamental identities.

Lemma 4.1. *For any λ, $\mu \in \mathrm{Spec}(\mathcal{H})$, if $\lambda \neq \mu$ then*

(a) $(\lambda - \mu)g(\nabla_X Y, V) = g((\nabla_X A)Y, V)$

(b) $\nabla_Y Z \perp T_\mu$

(c) $g(\nabla_{[Y,V]}Y, V) = g(\nabla_Y V, \nabla_V Y) + c\big(g(\phi Y, V)^2 + \sum_{a=1}^3 \{g(\phi_a Y, V)^2 + g(\theta_a Y, V)^2\}\big)$

for all vector fields Y, Z tangent to T_λ, V tangent to T_μ, and $X \in TM$.

Proof. Statement (a) is trivial. Next, by the Codazzi equation and (a), we have

$$\begin{aligned} 0 &= g((\nabla_Y A)V, Z) - g((\nabla_V A)Y, Z) \\ &= (\lambda - \mu)g(\nabla_Y Z, V). \end{aligned}$$

This gives Statement (b). Similarly, with the help of the Codazzi equation and (2.1), we compute

$$
\begin{aligned}
&(\mu-\lambda)g(\nabla_{[Y,V]}V,Y)\\
&\quad= g((\nabla_{[Y,V]}A)V,Y)\\
&\quad= g((\nabla_{\nabla_Y V}A)V,Y)-g((\nabla_{\nabla_V Y}A)Y,V)\\
&\quad= g((\nabla_V A)Y,\nabla_Y V)-g((\nabla_Y A)V,\nabla_V Y)\\
&\qquad+c\Big(\eta(\nabla_Y V)g(\phi V,Y)+\sum_{a=1}^{3}\{\eta_a(\nabla_Y V)g(\phi_a V,Y)+\eta(\phi_a\nabla_Y V)g(\theta_a V,Y)\}\Big)\\
&\qquad-c\Big(\eta(\nabla_V Y)g(\phi Y,V)+\sum_{a=1}^{3}\{\eta_a(\nabla_V Y)g(\phi_a Y,V)+\eta(\phi_a\nabla_V Y)g(\theta_a Y,V)\}\Big)\\
&\quad=(\lambda-\mu)g(\nabla_V Y,\nabla_Y V)\\
&\qquad+(\lambda-\mu)c\Big(g(\phi Y,V)^2+\sum_{a=1}^{3}\{g(\phi_a Y,V)^2+g(\theta_a Y,V)^2\}\Big).
\end{aligned}
$$

Hence we obtain Statement (c). □

If $\sqrt{-c}\notin\mathrm{Spec}(\mathcal{H})$, then M is an open part of a real hypersurface of type A, B or C_1 in view of [6, Theorem 3.4]. Next, consider the case $\sqrt{-c}\in\mathrm{Spec}(\mathcal{H})$. We claim that $\mathrm{Spec}(\mathcal{H})=\{\sqrt{-c}\}$. Suppose to the contrary that $\mathrm{Spec}(\mathcal{H})\neq\{\sqrt{-c}\}$. Let $\dim T_{\sqrt{-c}}=4m-8-p$, $p>0$, and let $\{X_1,\dots,X_p\}$ be a local orthonormal frame on $\mathcal{H}$ with $AX_j=\lambda_j X_j$, where $\lambda_j\neq\sqrt{-c}$ for each $j\in\{1,\dots,p\}$. Taking a unit vector field E tangent to $T_{\sqrt{-c}}$, we have

$$
\begin{aligned}
&c\Big(3g(\phi E,X_j)^2+\sum_{a=1}^{3}\{3g(\phi_a E,X_j)^2-g(\theta_a E,X_j)^2+g(\theta_a E,E)g(\theta_a X_j,X_j)\}\Big)\\
&\qquad+c+\lambda_j\sqrt{-c}\\
&\quad=g(R(X_j,E)E,X_j)\\
&\quad=g(\nabla_{X_j}\nabla_E E,X_j)-g(\nabla_E\nabla_{X_j}E,X_j)-g(\nabla_{[X_j,E]}E,X_j)\\
&\quad=-g(\nabla_E E,\nabla_{X_j}X_j)+g(\nabla_{X_j}E,\nabla_E X_j)-g(\nabla_{[X_j,E]}E,X_j)\\
&\quad=-\lambda\sqrt{-c}\sum_{a=1}^{3}g(\theta_a E,E)g(\theta_a X_j,X_j)+2g(\nabla_{X_j}E,\nabla_E X_j)\\
&\qquad+c\Big(g(\phi E,X_j)^2+\sum_{a=1}^{3}\{g(\phi_a E,X_j)^2+g(\theta_a E,X_j)^2\}\Big)\\
&\quad=-\lambda\sqrt{-c}\sum_{a=1}^{3}g(\theta_a E,E)g(\theta_a X_j,X_j)+2\sum_{\substack{k=1\\ \lambda_k\neq\lambda_j}}^{p}g(\nabla_{X_j}E,X_k)g(\nabla_E X_j,X_k)
\end{aligned}
$$

$$- 2\lambda\sqrt{-c}\Big(g(\phi E, X_j)^2 + \sum_{a=1}^{3}\{g(\phi_a E, X_j)^2 - g(\theta_a E, X_j)^2\}\Big)$$
$$+ c\Big(g(\phi E, X_j)^2 + \sum_{a=1}^{3}\{g(\phi_a E, X_j)^2 + g(\theta_a E, X_j)^2\}\Big).$$

It follows that

$$\begin{aligned}
&(c+\lambda_j\sqrt{-c})\Big(1 + 2g(\phi E, X_j)^2 \\
&\qquad + \sum_{a=1}^{3}\{2g(\phi_a E, X_j)^2 - 2g(\theta_a E, X_j)^2 + g(\theta_a E, E)g(\theta_a X_j, X_j)\}\Big) \\
&= 2\sum_{\substack{k=1\\ \lambda_k\neq\lambda_j}}^{p} g(\nabla_{X_j}E, X_k)g(\nabla_E X_j, X_k) \\
&= 2\sum_{\substack{k=1\\ \lambda_k\neq\lambda_j}}^{p} \frac{g((\nabla_{X_j}A)E, X_k)}{\sqrt{-c}-\lambda_k}\frac{g((\nabla_E A)X_j, X_k)}{\lambda_j-\lambda_k} \\
&= 2\sum_{\substack{k=1\\ \lambda_k\neq\lambda_j}}^{p} \frac{g((\nabla_{X_k}A)E, X_j)^2}{(\sqrt{-c}-\lambda_k)(\lambda_j-\lambda_k)}.
\end{aligned}$$

By applying the Codazzi equation and the preceding equation, we have

$$\begin{aligned}
0 &= 2\sum_{j=1}^{p}\sum_{\substack{k=1\\ \lambda_k\neq\lambda_j}}^{p} \frac{g((\nabla_{X_k}A)E, X_j)^2}{(\lambda_j-\sqrt{-c})(\sqrt{-c}-\lambda_k)(\lambda_j-\lambda_k)} \\
&= \sum_{j=1}^{p}\frac{c+\lambda_j\sqrt{-c}}{\lambda_j-\sqrt{-c}}\Big(1 + 2g(\phi E, X_j)^2 \\
&\quad + \sum_{a=1}^{3}\{2g(\phi_a E, X_j)^2 - 2g(\theta_a E, X_j)^2 + g(\theta_a E, E)g(\theta_a X_j, X_j)\}\Big) \qquad (4.1)\\
&= \sum_{j=1}^{p}\Big(1 + 2g(\phi E, X_j)^2 \\
&\quad + \sum_{a=1}^{3}\{2g(\phi_a E, X_j)^2 - 2g(\theta_a E, X_j)^2 + g(\theta_a E, E)g(\theta_a X_j, X_j)\}\Big).
\end{aligned}$$

By Theorem 1.2, we know that $\phi T_{\lambda_j} \subset T_{\sqrt{-c}}$ and $\phi_a T_{\lambda_j} \subset T_{\sqrt{-c}}$. If we take $E = \phi X_1$, then $\theta_a E = -\phi_a X_1$ is tangent to $T_{\sqrt{-c}}$. Hence, equation (4.1) reduces to

$$0 = p + 2 + \sum_{j=1}^{p}\sum_{a=1}^{3}\left\{2g(\theta_a X_1, X_j)^2 + g(\theta_a X_1, X_1)g(\theta_a X_j, X_j)\right\}. \qquad (4.2)$$

For fixed $b \in \{1,2,3\}$, by substituting $E = \phi_b X_1$ in (4.1) we obtain

$$\begin{aligned} 0 = p + 2 + \sum_{j=1}^{p} \big\{ & 2g(\theta_b X_1, X_j)^2 - 2g(\theta_{b+1} X_1, X_j)^2 - 2g(\theta_{b+2} X_1, X_j)^2 \\ & + g(\theta_b X_1, X_1) g(\theta_b X_j, X_j) - g(\theta_{b+1} X_1, X_1) g(\theta_{b+1} X_j, X_j) \\ & - g(\theta_{b+2} X_1, X_1) g(\theta_{b+2} X_j, X_j) \big\}. \end{aligned}$$

By summing over b, we obtain

$$0 = 3p + 6 - \sum_{j=1}^{p} \sum_{b=1}^{3} \big\{ 2g(\theta_b X_1, X_j)^2 + g(\theta_b X_1, X_1) g(\theta_b X_j, X_j) \big\}. \tag{4.3}$$

Adding (4.2) and (4.3) gives $4p + 8 = 0$; a contradiction. Accordingly, $\mathrm{Spec}(\mathcal{H}) = \{\sqrt{-c}\}$ and hence M is an open part of a real hypersurface of type C_2.

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