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COMPLETE LIFTING OF DOUBLE-LINEAR SEMI-BASIC TANGENT VALUED FORMS TO WEIL LIKE FUNCTORS ON DOUBLE VECTOR BUNDLES

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ABSTRACT. Let F be a product preserving gauge bundle functor on double vector bundles. We introduce the complete lifting $\mathcal{F}\varphi : FK \to \wedge^p T^*FM \otimes TFK$ of a double-linear semi-basic tangent valued p-form $\varphi : K \to \wedge^p T^*M \otimes TK$ on a double vector bundle K with base M. We prove that this complete lifting preserves the Frolicher–Nijenhuis bracket. We apply the results obtained to double-linear connections.

1. Introduction

We assume that any manifold considered in the paper is Hausdorff, second countable, finite dimensional, without boundary and smooth (i.e. of class C^∞). All maps between manifolds are assumed to be smooth (of class C^∞).

Definition 1.1. An *almost double vector bundle* is a system $K = (K_r, K_l, E_r, E_l)$ of vector bundles $K_r = (K, \tau_r, E_r)$, $K_l = (K, \tau_l, E_l)$, $E_r = (E_r, \underline{\tau}_l, M)$ and $E_l = (E_l, \underline{\tau}_r, M)$ such that $\underline{\tau}_l \circ \tau_r = \underline{\tau}_r \circ \tau_l$ (this means that the respective diagram is commutative). We call M the *basis* of K.

If $K' = (K'_r, K'_l, E'_r, E'_l)$ is another almost double vector bundle, an almost double vector bundle map $K \to K'$ is a map $f : K \to K'$ such that there are maps $\underline{f}_r : E_r \to E'_r$, $\underline{f}_l : E_l \to E'_l$ and $\underline{f} : M \to M'$ such that $(f, \underline{f}_r) : K_r \to K'_r$, $(f, \underline{f}_l) : K_l \to K'_l$, $(\underline{f}_r, \underline{f}) : E_r \to E'_r$ and $(\underline{f}_l, \underline{f}) : E_l \to E'_l$ are vector bundle maps. We call $\underline{f} : M \to M'$ the base map of f.

For example, we have the trivial almost double vector bundle $K = (K_r, K_l, E_r, E_l)$, where $K_l = (\mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}, \tau_l, \mathbf{R}^{m_1} \times \mathbf{R}^{n_1})$, $K_r = (\mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}, \tau_r, \mathbf{R}^{m_1} \times \mathbf{R}^{m_2})$, $E_r = (\mathbf{R}^{m_1} \times \mathbf{R}^{m_2}, \underline{\tau}_l, \mathbf{R}^{m_1})$ and $E_l = (\mathbf{R}^{m_1} \times \mathbf{R}^{n_1}, \underline{\tau}_r, \mathbf{R}^{m_1})$, and where τ_r, τ_l, $\underline{\tau}_r$, $\underline{\tau}_l$ are the obvious projections. We will denote this trivial almost double vector bundle by $\mathbf{R}^{m_1,m_2,n_1,n_2}$.

Definition 1.2. A *double vector bundle* is a locally trivial almost double vector bundle K. This means that there are nonnegative integers m_1, m_2, n_1, n_2 such

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that for any $x \in M$ there is an open neighborhood $\Omega \subset M$ of x such that $K_{|\Omega} = \mathbf{R}^{m_1,m_2,n_1,n_2}$ modulo an almost double vector bundle isomorphism.

The tangent bundle

$$TE = ((TE, \pi^{TE}, E), (TE, T\pi, TM), (E, \pi, M), (TM, \pi^{TM}, M))$$

of a vector bundle $E = (E, \pi, M)$ is an example of a double vector bundle.

Any manifold M can be treated as the double vector bundle M with basis M.

Definition 1.3. Let K be a double vector bundle as above. A *double-linear vector field* on K is a vector field Z on K such that the flow of Z is formed by (local) double vector bundle isomorphisms.

Any double linear vector field Z on K is projectable with respect to the (common) projection $K \to M$. Thus we have the underlying vector field $\underline{Z}$ on M.

Definition 1.4. Let K be a double vector bundle as above with basis M. A *double-linear semi-basic tangent valued p-form* on K is a section $\varphi : K \to \wedge^p T^*M \otimes TK$ such that $\varphi(X_1, \dots, X_p)$ is a double linear vector field on K for any vector fields $X_1, \dots, X_p$ on the basis M of K.

Definition 1.5. Let K be as above. A *double-linear connection* in K is a double-linear semi-basic tangent valued 1-form $\Gamma : K \to T^*M \otimes TK$ on K such that the underlying vector field of $\Gamma(X)$ is equal to X for any vector field X on basis M.

Let $\mathcal{DVB}$ denote the category of all double vector bundles and their almost double vector bundle maps, and let $\mathcal{FM}$ denote the category of fibered manifolds and fibered maps. (In [14], the notation 2-$\mathcal{VB}$ instead of $\mathcal{DVB}$ is used.)

The general concept of (gauge) bundle functors can be found in [7]. We need the following particular case of it.

Definition 1.6. A *gauge bundle functor* on $\mathcal{DVB}$ is a covariant functor $F : \mathcal{DVB} \to \mathcal{FM}$ sending any double vector bundle K with basis M into a fibered manifold $p_K : FK \to M$ over M and any double vector bundle map $f : K \to K'$ with the base map $\underline{f} : M \to M'$ into a fibered map $Ff : FK \to FK'$ over $\underline{f} : M \to M'$, and satisfying the following conditions:

(i) *Localization condition*: For every double vector bundle K with basis M and any open subset $U \subset M$ the inclusion map $i_{K|U} : K|U \to K$ induces a diffeomorphism $Fi_{K|U} : F(K|U) \to p_K^{-1}(U)$.

(ii) *Regularity condition*: F transforms smoothly parametrized families of $\mathcal{DVB}$-maps into smoothly parametrized families of $\mathcal{FM}$-maps.

A gauge bundle functor F on $\mathcal{DVB}$ is product preserving (ppgb-functor) if $F(K_1 \times K_2) = F(K_1) \times F(K_2)$ for any $\mathcal{DVB}$-objects K_1 and K_2. Product preserving gauge bundle functors can be also called Weil like functors, because the product preserving bundle functors on manifolds are the usual Weil functors.

A simple example of a ppgb-functor on $\mathcal{DVB}$ is the tangent functor T sending any $\mathcal{DVB}$-object K into the tangent bundle TK (over M) and any $\mathcal{DVB}$-map $f : K \to K'$ into the tangent map $Tf : TK \to TK'$.

By [14], the ppgb-functors F on $\mathcal{DVB}$ are in bijection with the A^F-bilinear maps $\diamond^F : U^F \times V^F \to W^F$, where A^F are Weil algebras and U^F, V^F and W^F are finitely dimensional (over $\mathbf{R}$) A^F-modules. Moreover, the ppgb-functors F on $\mathcal{DVB}$ have values in $\mathcal{DVB}$. For any such F, if K is a $\mathcal{DVB}$-object with basis M, then FK is a $\mathcal{DVB}$-object with basis $FM = T^{A^F}M$; see [14].

Let F be a ppgb-functor on $\mathcal{DVB}$ and let $\diamond^F : U^F \times V^F \to W^F$ be the corresponding A^F-bilinear map. Let K be a $\mathcal{DVB}$-object. Then any double-linear vector field Z on K can be lifted to the double-linear vector field $\mathcal{F}Z$ on FK via F-prolongation of flow. By [14], for any $a \in A^F$ we have the affinor $\mathrm{af}(a) : TFK \to TFK$ on FK. We have $\mathrm{af}(a_1a_2) = \mathrm{af}(a_1) \circ \mathrm{af}(a_2)$ and $\mathrm{af}(1)$ is the identity affinor. If $f : K \to K_1$ is a $\mathcal{DVB}$-map, then $TFf \circ \mathrm{af}(a) = \mathrm{af}(a) \circ TFf$.

The main result of the paper is the following one (see Theorem 4.5):

Let F be a ppgb-functor on $\mathcal{DVB}$. Let $\varphi : K \to \wedge^p T^*M \otimes TK$ be a double-linear semi-basic tangent valued p-form on a double vector bundle K with basis M. Then there exists one and only one double-linear semi-basic tangent valued p-form $\mathcal{F}\varphi : FK \to \wedge^p T^*FM \otimes TFK$ on FK such that

$$\mathcal{F}\varphi(\mathrm{af}(a_1) \circ \mathcal{F}X_1, \dots, \mathrm{af}(a_p) \circ \mathcal{F}X_p) = \mathrm{af}(a_1 \cdot \dots \cdot a_p) \circ \mathcal{F}(\varphi(X_1, \dots, X_p))$$

for any vector fields $X_1, \dots, X_p$ on M and any $a_1, \dots, a_p \in A^F$.

Definition 1.7. We call $\mathcal{F}\varphi$ (as above) the *complete lift* of φ to F.

Next we study the complete lifting $\mathcal{F}$. We prove that $\mathcal{F}$ commutes with the Frolicher–Nijenhuis bracket (see Theorem 5.1) and apply this fact to double-linear connections $\Gamma : K \to T^*M \otimes TK$ in K (see Theorem 6.3).

By the local description of double vector bundles, presented in [8], the notion of double vector bundles in the sense of the present paper is equivalent to the one in the book [11]. Product preserving (gauge) bundle functors are studied in [1, 6, 7, 9, 10, 12, 13, 14, 16, 17, 18]. Liftings of vector fields to product preserving (gauge) bundle functors are studied in [5, 10, 14]. Complete lifting of general connections on fibered manifolds to Weil functors is studied in [7]. Complete lifting of semi-basic tangent valued p-forms on fibered manifolds to Weil functors is studied in [2, 3]. Complete lifting of linear semi-basic tangent valued forms to product preserving gauge bundle functors on vector bundles is studied in [15]. The Frolicher–Nijenhuis bracket on projectable tangent valued forms is studied in [4].

2. Preliminaries

Let K be a double vector bundle. Let M be the basis of K and $\pi : K \to M$ be the projection.

Lemma 2.1. *Let Z, Z_1 be double-linear vector fields on K, α a real number and $f : M \to \mathbf{R}$ a map. Then $Z + Z_1$, αZ, $f \circ \pi \cdot Z$ and $[Z, Z_1]$ are double linear vector fields on K.*

Proof. Using $\mathcal{DVB}$-charts, we may assume $K = \mathbf{R}^{m_1,m_2,n_1,n_2}$. Let $x^1, \dots, x^{m_1}$, $u^1, \dots, u^{m_2}$, $v^1, \dots, v^{n_1}$, $w^1, \dots, w^{n_2}$ be the usual coordinates. A map $f : K \to K$

is a $\mathcal{DVB}$-map if and only if it is of the form

$$x^i \circ f = \alpha^i(x), \quad i = 1, \dots, m_1,$$
$$u^j \circ f = \sum_{j_1=1}^{m_2} \beta^j_{j_1}(x) u^{j_1}, \quad j = 1, \dots, m_2,$$
$$v^k \circ f = \sum_{k_1=1}^{n_1} \gamma^k_{k_1}(x) v^{k_1}, \quad k = 1, \dots, n_1,$$
$$w^l \circ f = \sum_{l_1=1}^{n_2} \gamma^l_{l_1}(x) w^{l_1} + \sum_{j_1=1}^{m_2} \sum_{k_1=1}^{n_1} \sigma^l_{j_1 k_1}(x) u^{j_1} v^{k_1}, \quad l = 1, \dots, n_2,$$

where $x = (x^1, \dots, x^{m_1})$. Consequently, a vector field Z on K is double linear if and only if it is of the form

$$\begin{aligned} Z = \sum_{i=1}^{m_1} a^i(x) \frac{\partial}{\partial x^i} + \sum_{j,j^1=1}^{m_2} b^{j_1}_j(x) u^j \frac{\partial}{\partial u^{j_1}} + \sum_{k,k_1=1}^{n_1} c^{k_1}_k(x) v^k \frac{\partial}{\partial v^{k_1}} \\ + \sum_{l,l^1=1}^{n_2} e^{l_1}_l(x) w^l \frac{\partial}{\partial w^{l_1}} + \sum_{j_2=1}^{m_2} \sum_{k_2=1}^{n_1} \sum_{l_2=1}^{n_2} f^{l_2}_{j_2 k_2}(x) u^{j_2} v^{k_2} \frac{\partial}{\partial w^{l_2}}. \end{aligned} \tag{2.1}$$

The lemma is now clear. □

Now, we treat K as a fibered manifold over M or (generally) let $\pi : K \to M$ be an arbitrary fibered manifold.

Definition 2.2. A *projectable semi-basic tangent valued p-form* on K is a section $\varphi : K \to \wedge^p T^*M \otimes TK$ such that $\varphi(X_1, \dots, X_p)$ is a projectable vector field on K.

Given a projectable semi-basic tangent valued p-form $\varphi : K \to \wedge^p T^*M \otimes TK$ we have the underlying tangent valued p-form $\underline{\varphi} : M \to \wedge^p T^*M \otimes TM$ on M such that $\underline{\varphi}(X_1, \dots, X_p)$ is the underlying vector field of the projectable vector field $\varphi(X_1, \dots, X_p)$ for any vector fields $X_1, \dots, X_p$ on M.

The following fact is well known; see e.g. [3, 4].

Lemma 2.3. *Given a projectable semi-basic tangent-valued p-form $\varphi : K \to \wedge^p T^*M \otimes TK$ on K and a projectable semi-basic tangent valued q-form $\psi : K \to \wedge^q T^*M \otimes TK$ on K there exists a (unique) projectable semi-basic tangent valued*

$(p+q)$-form $[[\varphi,\psi]] : K \to \wedge^{p+q}T^*M \otimes TK$ on K such that

$$
\begin{aligned}
&[[\varphi,\psi]](X_1,\dots,X_{p+q})\\
&\quad = \frac{1}{p!q!}\sum_\sigma \operatorname{sgn}\sigma\cdot[\varphi(X_{\sigma 1},\dots,X_{\sigma p}),\psi(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})]\\
&\qquad + \frac{-1}{p!(q-1)!}\sum_\sigma \operatorname{sgn}\sigma\cdot\psi([\underline{\varphi}(X_{\sigma 1},\dots,X_{\sigma p}),X_{\sigma(p+1)}],X_{\sigma(p+2)},\dots)\\
&\qquad + \frac{(-1)^{pq}}{(p-1)!q!}\sum_\sigma \operatorname{sgn}\sigma\cdot\varphi([\underline{\psi}(X_{\sigma 1},\dots,X_{\sigma q}),X_{\sigma(q+1)}],X_{\sigma(q+2)},\dots) \qquad (2.2)\\
&\qquad + \frac{(-1)^{p-1}}{(p-1)!(q-1)!2!}\sum_\sigma \operatorname{sgn}\sigma\cdot\psi(\underline{\varphi}([X_{\sigma 1},X_{\sigma 2}],X_{\sigma 3},\dots),X_{\sigma(p+2)},\dots)\\
&\qquad + \frac{(-1)^{(p-1)q}}{(p-1)!(q-1)!2!}\sum_\sigma \operatorname{sgn}\sigma\cdot\varphi(\underline{\psi}([X_{\sigma_1},X_{\sigma 2}],X_{\sigma 3},\dots),X_{\sigma(q+2)},\dots)
\end{aligned}
$$

for any vector fields $X_1,\dots,X_{p+q}$ on M, where sums are over all permutations $\sigma : \{1,\dots,p+q\}\to\{1,\dots,p+q\}$ and $\operatorname{sgn}\sigma$ is the signum of σ.

The underlying tangent valued $(p+q)$-form of $[[\varphi,\psi]]$ is $[[\underline{\varphi},\underline{\psi}]]$.

Definition 2.4. The bracket $[[-,-]]$ is called the *Frolicher–Nijenhuis bracket.*

Proposition 2.5. *Let K be a double vector bundle with basis M. Let $\varphi : K \to \wedge^p T^*M\otimes TK$ be a double-linear (then projectable) semi-basic tangent valued p-form on K and let $\psi : K \to \wedge^q T^*M \otimes TK$ be a double-linear semi-basic tangent valued q-form on K. Then the Frolicher–Nijenhuis bracket $[[\varphi,\psi]] : K \to \wedge^{p+q}T^*M\otimes TK$ is a double-linear semi-basic tangent valued $(p+q)$-form on K.*

Proof. It follows from formula (2.2), Lemma 2.1 and Definition 1.4. □

We end this section with the $\mathcal{DVB}$-version of the well-known fact of the simplicity of vector fields.

Lemma 2.6. *Let Z be a double linear vector field on a double vector bundle K such that the underlying vector field $\underline{Z}$ on basis M is nonzero at a point $x_o \in M$. Then there exists a local $\mathcal{DVB}$-coordinate system $(x^1,\dots)$ on K with centrum x_o such that $Z = \frac{\partial}{\partial x^1}$.*

Proof. The proof is quite similar to that of the manifold case. We may assume that $K = \mathbf{R}^{m_1,m_2,n_1,n_2}$, $x_o = 0$ and $\underline{Z}_{|0} = \frac{\partial}{\partial x^1}_{|0}$. Let $\{\varphi_t\}$ be the flow of Z. Then $\Phi : K \to K$ given by $\Phi(x^1,\dots) = \varphi_{x_1}(0,x^2,\dots)$ is a local $\mathcal{DVB}$-isomorphism sending $\frac{\partial}{\partial x^1}$ to Z. □

3. On the complete lifting of double-linear vector fields to ppgb-functors on double vector bundles

Let $F : \mathcal{DVB} \to \mathcal{FM}$ be a ppgb-functor. We know that $F : \mathcal{DVB} \to \mathcal{DVB}$. Let Z be a double-linear vector field on a double vector bundle K.

Definition 3.1. The *complete lift* of Z to F is the double-linear vector field $\mathcal{F}Z$ on FK corresponding to the flow $\{F\varphi_t\}$, where $\{\varphi_t\}$ is the flow of Z.

Lemma 3.2. *If $\varphi : K \to K_1$ is a (locally defined) $\mathcal{DVB}$-isomorphism, then $\mathcal{F}(\varphi_* Z) = (F\varphi)_* \mathcal{F}Z$.*

Proof. The flow of $\varphi_* Z$ is $\{\varphi \circ \varphi_t \circ \varphi^{-1}\}$. Then the flow of $\mathcal{F}(\varphi_* Z)$ is $\{F\varphi \circ F\varphi_t \circ (F\varphi)^{-1}\}$. The last flow is the one of $(F\varphi)_* \mathcal{F}Z$. □

Lemma 3.3. *If α is a real number, then $\mathcal{F}(\alpha Z) = \alpha \mathcal{F}Z$. Consequently, $\mathcal{F}(\alpha Z + \alpha_1 Z_1) = \alpha FZ + \alpha_1 \mathcal{F}Z_1$ for any real numbers α and α_1 and any double linear vector fields Z and Z_1 on K.*

Proof. If $\{\varphi_t\}$ is the flow of Z, then $\{\varphi_{\alpha t}\}$ is the flow of αZ. So, $\{F\varphi_{\alpha t}\}$ is the flow of $\mathcal{F}(\alpha Z)$ and of $\alpha \mathcal{F}Z$. Hence, $\mathcal{F}$ is $\mathbf{R}$-linear because of the homogeneous function theorem and the nonlinear Peetre theorem [7]. □

Let $\diamond^F : U^F \times V^F \to W^F$ be the A^F-bilinear map corresponding to F.

Lemma 3.4. *Let Z be a double linear vector field on a double vector bundle K with basis M and let $a \in A^F$. Then $\mathrm{af}(a) \circ \mathcal{F}Z$ is a double linear vector field on FK.*

Proof. We may assume that the underlying vector field $\underline{Z}$ is nowhere vanishing. Then using $\mathcal{DVB}$-charts and Lemma 2.6 we may assume that $Z = \frac{\partial}{\partial x^1}$ and $K = \mathbf{R}^{m_1,m_2,n_1,n_2}$. Then $FK = (A^F)^{m_1} \times (U^F)^{m_2} \times (V^F)^{n_1} \times (W^F)^{n_2}$ and $\mathrm{af}(a) \circ \mathcal{F}Z$ can be treated as a vector field on $(A^F)^{m_1}$ (and consequently as a double linear vector field on FK). □

By Lemma 2.1, if Z and Z_1 are double linear vector fields on K then so is $[Z, Z_1]$.

Proposition 3.5. *For any double linear vector fields Z and Z_1 on K and any $a, a_1 \in A^F$ we have*

$$[\mathrm{af}(a) \circ \mathcal{F}Z, \mathrm{af}(a_1) \circ \mathcal{F}Z_1] = \mathrm{af}(aa_1) \circ \mathcal{F}([Z, Z_1]). \tag{3.1}$$

Proof. We may assume that $K = \mathbf{R}^{m_1,m_2,n_1,n_2}$, $Z = \frac{\partial}{\partial x^1}$ and $Z_1 = f(x^1, \dots, x^{m_1}) Z_2$, where $Z_2 \in \left\{\frac{\partial}{\partial x^i}, u^j \frac{\partial}{\partial u^{j_1}}, v^k \frac{\partial}{\partial v^{k_1}}, w^l \frac{\partial}{\partial w^{l_1}}, u^j v^k \frac{\partial}{\partial w^l}\right\}$.

If $Z_2 = \frac{\partial}{\partial x^i}$, then the formula is the well-know one for usual Weil functors on manifolds. For other values of Z_2, using formula (3.2) (below) and the known formula $a\mathcal{F}Z(a_1 Ff) = aa_1 F(Z(f))$ for usual Weil functors on manifolds, we get $[\mathrm{af}(a) \circ \mathcal{F}Z, \mathrm{af}(a_1) \circ \mathcal{F}(fZ_2)] = [a \cdot \mathcal{F}Z, a_1 Ff \cdot \mathcal{F}Z_2] = a\mathcal{F}Z(a_1 Ff) \cdot \mathcal{F}Z_2 = aa_1 F(Z(f)) \cdot \mathcal{F}Z_2 = aa_1 \cdot \mathcal{F}(Z(f) Z_2) = \mathrm{af}(aa_1) \circ \mathcal{F}([Z, Z_1])$. □

Lemma 3.6. *Let Z be a double linear vector field on K and let $f : M \to \mathbf{R}$ be a map. Then*

$$\mathcal{F}(f \circ \pi \cdot Z) = Ff \circ F\pi \cdot \mathcal{F}Z, \tag{3.2}$$

where $\pi : K \to M$ is the projection (we treat M as a $\mathcal{DVB}$-object and π as a $\mathcal{DVB}$-map in the obvious way) and $Ff : FM \to F\mathbf{R} = A^F$. Here (in the right of the formula) $a \cdot y := \mathrm{af}(a)(y)$ for $a \in A^F$ and $y \in TFK$.

Proof. By Lemma 2.1, $f\circ\pi\cdot Z$ is double linear. So, both sides of (3.2) make sense. By the linearity of $\mathcal{F}$, we may assume that Z is not π-vertical. Then by Lemma 2.6 we may assume that $K=\mathbf{R}^{m_1,m_2,n_1,n_2}$ and $Z=\frac{\partial}{\partial x^1}$. Then we may additional assume that $K=M$ is a manifold, Z is a vector field on M and F is a Weil functor on manifolds. Then our lemma is the (well known for Weil functors on manifolds) formula $\mathcal{F}(fZ)=Ff\cdot\mathcal{F}Z$. □

4. On the complete lifting of double-linear semi-basic tangent valued p-forms to ppgb-functors on double vector bundles

For a moment, let F be a ppgb-functor (Weil functor) on manifolds. Let $\omega\in\Omega^p(M)$ be a p-form on a manifold M. Then $\omega: TM\times_M\ldots\times_M TM\to\mathbf{R}$ is a fiber skew p-linear map. Applying F, we get the fibre skew p-linear (over A^F) map $F\omega: FTM\times_{FM}\ldots\times_{FM}FTM\to A^F$ (this is a well-known fact for Weil functors on manifolds). Then applying the exchange isomorphism $\eta_M: TFM\to FTM$, which is a vector bundle isomorphism (this is also a well-known fact for Weil functors on manifolds), we obtain the A^F-valued p-form

$$\mathcal{F}\omega:=F\omega\circ(\eta_M\times\ldots\times\eta_M): TFM\times_{FM}\ldots\times_{FM}TFM\to A^F$$

over FM.

Lemma 4.1. *$\mathcal{F}\omega$ is the unique A^F-valued p-form on FM such that*

$$\mathcal{F}\omega(\mathrm{af}(a_1)\circ\mathcal{F}X_1,\ldots,\mathrm{af}(a_p)\circ\mathcal{F}X_p)=a_1\cdot\ldots\cdot a_p\cdot F(\omega(X_1,\ldots,X_p))\qquad(4.1)$$

for any vector fields $X_1,\ldots,X_p$ on M and any $a_1,\ldots,a_p\in A^F$.

Proof. The uniqueness is a consequence of the well-known fact for Weil functors on manifolds that the vector fields $\mathrm{af}(a)\circ\mathcal{F}X$ generate over $C^\infty(M)$ the vector space $\mathcal{X}(FM)$. Formula (4.1) follows from the well-known (for Weil functors on manifolds) equalities $\mathcal{F}X=\eta_M^{-1}\circ FX$ and $\eta_M\circ\mathrm{af}(a)=a\cdot\eta_M$. □

Definition 4.2. The A^F-valued p-form on FM satisfying (4.1) is called the *complete lift* of ω to F.

For the rest of this section, let $F:\mathcal{DVB}\to\mathcal{FM}$ be a ppgb-functor.

Let $x^1,\ldots,x^{m_1}$, $u^1,\ldots,u^{m_2}$, $v^1,\ldots,v^{n_1}$, $w^1,\ldots,w^{n_2}$ be the usual coordinates on $\mathbf{R}^{m_1,m_2,n_1,n_2}$.

Because of the local expression (2.1) of double-linear vector fields and of the Definition 1.4 of double-linear semi-basic tangent valued p-forms, any double-linear semi-basic tangent valued p-form φ on $\mathbf{R}^{m_1,m_2,n_1,n_2}$ is of the form

$$\begin{aligned}\varphi=&\sum_{i=1}^{m_1}\varphi^i\otimes_{\mathbf{R}}\frac{\partial}{\partial x^i}+\sum_{j,j_1=1}^{m_2}\psi^j_{j_1}\otimes_{\mathbf{R}}u^{j_1}\frac{\partial}{\partial u^j}\\&+\sum_{k,k_1=1}^{n_1}\chi^k_{k_1}\otimes_{\mathbf{R}}v^{k_1}\frac{\partial}{\partial v^k}+\sum_{l,l_1=1}^{n_2}\xi^l_{l_1}\otimes_{\mathbf{R}}w^{l_1}\frac{\partial}{\partial w^l}\\&+\sum_{j=1}^{m_2}\sum_{k=1}^{n_1}\sum_{l=1}^{n_2}\rho^l_{jk}\otimes_{\mathbf{R}}u^jv^k\frac{\partial}{\partial w^l}\end{aligned}$$

for unique p-forms φ^i, $\psi^j_{j_1}$, $\chi^k_{k_1}$, $\xi^l_{l_1}$, ρ^l_{jk} on $\mathbf{R}^m$, where $(\omega \otimes_{\mathbf{R}} Z)(X_1, \dots, X_p) := \omega(X_1, \dots, X_p) \circ \pi \cdot Z$.

For any such φ we define its complete lift $\mathcal{F}\varphi$ by

$$\begin{aligned}\mathcal{F}\varphi := &\sum_{i=1}^{m_1} \mathcal{F}\varphi^i \otimes_{A^F} \mathcal{F}\frac{\partial}{\partial x^i} + \sum_{j,j_1=1}^{m_2} \mathcal{F}\psi^j_{j_1} \otimes_{A^F} \mathcal{F}\left(u^{j_1}\frac{\partial}{\partial u^j}\right) \\ &+ \sum_{k,k_1=1}^{n_1} \mathcal{F}\chi^k_{k_1} \otimes_{A^F} \mathcal{F}\left(v^{k_1}\frac{\partial}{\partial v^k}\right) + \sum_{l,l_1=1}^{n_2} \mathcal{F}\xi^l_{l_1} \otimes_{A^F} \mathcal{F}\left(w^{l_1}\frac{\partial}{\partial w^l}\right) \\ &+ \sum_{j=1}^{m_2}\sum_{k=1}^{n_1}\sum_{l=1}^{n_2} \mathcal{F}\rho^l_{jk} \otimes_{A^F} \mathcal{F}\left(u^j v^k \frac{\partial}{\partial w^l}\right),\end{aligned} \tag{4.2}$$

where $(\mathcal{F}\omega \otimes_{A^F} \mathcal{F}Z)(Y_1, \dots, Y_p) := \mathcal{F}\omega(Y_1, \dots, Y_p) \circ F\pi \cdot \mathcal{F}Z$ for $Y_1, \dots, Y_p \in \mathcal{X}(F\mathbf{R}^{m_1})$.

Proposition 4.3. *The complete lift $\mathcal{F}\varphi$ as in* (4.2) *is the unique double-linear semi-basic tangent valued p-form on $F\mathbf{R}^{m_1,m_2,n_1,n_2}$ such that*

$$\mathcal{F}\varphi(\mathrm{af}(a_1) \circ \mathcal{F}X_1, \dots, \mathrm{af}(a_p) \circ \mathcal{F}X_p) = \mathrm{af}(a_1 \cdot \ldots \cdot a_p) \circ \mathcal{F}(\varphi(X_1, \dots, X_p)) \tag{4.3}$$

for any $a_1, \dots, a_p \in A^F$ and any $X_1, \dots, X_p \in \mathcal{X}(\mathbf{R}^{m_1})$.

Proof. The uniqueness is clear because the vector fields $\mathrm{af}(a) \circ \mathcal{F}X$ for $a \in A^F$ and $X \in \mathcal{X}(\mathbf{R}^{m_1})$ generate (over $C^\infty(F\mathbf{R}^{m_1})$) the vector space $\mathcal{X}(F\mathbf{R}^{m_1})$. This is a well-known fact for Weil functors on manifolds.

Now, we prove (4.3). Since both sides of (4.3) are linear in φ, we may assume that $\varphi = \omega \otimes_{\mathbf{R}} Z$, where $\omega \in \Omega^p(\mathbf{R}^{m_1})$ and $Z \in \left\{\frac{\partial}{\partial x^i}, u^{j_1}\frac{\partial}{\partial u^j}, v^{k_1}\frac{\partial}{\partial v^k}, w^{l_1}\frac{\partial}{\partial w^l}, u^j v^k \frac{\partial}{\partial w^l}\right\}$. Then by (4.2), (4.1) and (3.2) we have

$$\begin{aligned}&\mathcal{F}\varphi(\mathrm{af}(a_1) \circ \mathcal{F}X_1, \dots, \mathrm{af}(a_p) \circ \mathcal{F}X_p) \\ &\quad = \mathcal{F}(\omega \otimes_{\mathbf{R}} Z)(\mathrm{af}(a_1) \circ \mathcal{F}X_1, \dots, \mathrm{af}(a_p) \circ \mathcal{F}X_p) \\ &\quad = (\mathcal{F}\omega \otimes_{A^F} \mathcal{F}Z)(\mathrm{af}(a_1) \circ \mathcal{F}X_1, \dots, \mathrm{af}(a_p) \circ \mathcal{F}X_p) \\ &\quad = \mathcal{F}\omega(\mathrm{af}(a_1) \circ \mathcal{F}X_1, \dots, \mathrm{af}(a_p) \circ \mathcal{F}X_p) \circ F\pi \cdot \mathcal{F}Z \\ &\quad = a_1 \cdot \ldots \cdot a_p \cdot F(\omega(X_1, \dots, X_p)) \circ F\pi \cdot \mathcal{F}Z \\ &\quad = a_1 \cdot \ldots \cdot a_p \cdot \mathcal{F}(\omega(X_1, \dots, X_p) \circ \pi \cdot Z) \\ &\quad = a_1 \cdot \ldots \cdot a_p \cdot \mathcal{F}((\omega \otimes_{\mathbf{R}} Z)(X_1, \dots, X_p)) \\ &\quad = \mathrm{af}(a_1 \cdot \ldots \cdot a_p) \circ \mathcal{F}(\varphi(X_1, \dots, X_p)).\end{aligned}$$

□

Lemma 4.4. *For any (local) double vector bundle isomorphism $f : \mathbf{R}^{m_1,m_2,n_1,n_2} \to \mathbf{R}^{m_1,m_2,n_1,n_2}$ and any double-linear semi-basic tangent valued p-form φ on the double vector bundle $\mathbf{R}^{m_1,m_2,n_1,n_2}$, we have $(Ff)_*\mathcal{F}\varphi = \mathcal{F}(f_*\varphi)$.*

Proof. We have

$$\begin{aligned}
&(Ff)_*\mathcal{F}\varphi(\mathrm{af}(a_1)\circ\mathcal{F}X_1,\dots,\mathrm{af}(a_p)\mathcal{F}X_p)\\
&\quad=\mathcal{F}\varphi(Ff_*^{-1}(\mathrm{af}(a_1)\circ\mathcal{F}X_1),\dots,Ff_*^{-1}(\mathrm{af}(a_p)\circ\mathcal{F}X_p))\\
&\quad=\mathcal{F}\varphi(\mathrm{af}(a_1)\circ\mathcal{F}(f_*^{-1}X_1),\dots,\mathrm{af}(a_p)\circ\mathcal{F}(f_*^{-1}X_p))\\
&\quad=\mathrm{af}(a_1\cdot\ldots\cdot a_p)\circ\mathcal{F}\varphi(\mathcal{F}(f_*^{-1}X_1),\dots,\mathcal{F}(f_*^{-1}X_p))\\
&\quad=\mathrm{af}(a_1\cdot\ldots\cdot a_p)\circ\mathcal{F}(\varphi(f_*^{-1}X_1,\dots,f_*^{-1}X_p))\\
&\quad=\mathrm{af}(a_1\cdot\ldots\cdot a_p)\circ\mathcal{F}((f_*\varphi)(X_1,\dots,X_p))\\
&\quad=\mathcal{F}(f_*\varphi)(\mathrm{af}(a_1)\cdot\mathcal{F}X_1,\dots,\mathrm{af}(a_p)\cdot\mathcal{F}X_p).
\end{aligned}$$

Now, applying the uniqueness case of Proposition 4.3 (or, better, the sentence of the proof of the uniqueness case of Proposition 4.3) we end the proof. ☐

We are now in a position to prove the following result.

Theorem 4.5. *Let F be a ppgb-functor on $\mathcal{DVB}$. Let $\varphi : K \to \wedge^p T^*M \otimes TK$ be a double-linear semi-basic tangent valued p-form on a double vector bundle K with basis M. Then there exists one and only one double-linear semi-basic tangent valued p-form $\mathcal{F}\varphi : FK \to \wedge^p T^*FM \otimes TFK$ on FK such that*

$$\mathcal{F}\varphi(\mathrm{af}(a_1)\circ\mathcal{F}X_1,\dots,\mathrm{af}(a_p)\circ\mathcal{F}X_p)=\mathrm{af}(a_1\cdot\ldots\cdot a_p)\circ\mathcal{F}(\varphi(X_1,\dots,X_p)) \quad (4.4)$$

for any vector fields $X_1,\dots,X_p$ on M and any $a_1,\dots,a_p\in A^F$.

Proof. Using $\mathcal{DVB}$-charts on K, we spread the complete lifting of double-linear semi-basic tangent valued p-forms on $\mathbf{R}^{m_1,m_2,n_1,n_2}$ to the one on K. This is possible because of Lemma 4.4. ☐

5. The complete lifting of double-linear semi-basic tangent valued p-forms preserves the Frolicher–Nijenhuis bracket

Let F be a ppgb-functor on $\mathcal{DVB}$. Then $F : \mathcal{DVB} \to \mathcal{DVB}$.

Let $\varphi : K \to \wedge^p T^*M \otimes TK$ be a double-linear semi-basic tangent valued p-form on K and let $\psi : K \to \wedge^q T^*M \otimes TK$ be a double-linear semi-basic tangent valued q-form on K. We can lift φ and ψ to FK and obtain a double-linear semi-basic tangent valued p-form $\mathcal{F}\varphi$ on FK and a double-linear semi-basic tangent valued q-form $\mathcal{F}\psi$ on FK. Then we can produce the Frolicher–Nijenhuis bracket $[[\mathcal{F}\varphi,\mathcal{F}\psi]]$. By Proposition 2.5, this bracket is a double-linear semi-basic tangent valued $(p+q)$-form on FK.

On the other hand, by Proposition 2.5, the Frolicher–Nijenhuis bracket $[[\varphi,\psi]]$ is a double-linear semi-basic tangent valued $(p+q)$-form on K. So, we can lift it and obtain a double-linear semi-basic tangent valued $(p+q)$-form $\mathcal{F}([[\varphi,\psi]])$ on FK.

Theorem 5.1. *We have*

$$\mathcal{F}([[\varphi,\psi]])=[[\mathcal{F}\varphi,\mathcal{F}\psi]]. \quad (5.1)$$

Proof. For any $a_1, \dots, a_{p+1} \in A^F$ and vector fields $X_1, \dots, X_{p+q}$ on M we have

$$\begin{aligned}
&[\mathcal{F}\varphi(\mathrm{af}(a_1) \circ \mathcal{F}X_1, \dots, \mathrm{af}(a_p) \circ \mathcal{F}X_p),\\
&\qquad\qquad \mathcal{F}\psi(\mathrm{af}(a_{p+1}) \circ \mathcal{F}X_{p+1}, \dots, \mathrm{af}(a_{p+q}) \circ \mathcal{F}X_{p+q})]\\
&\qquad\qquad = \mathrm{af}(a_1 \cdot \ldots \cdot a_{p+q}) \circ \mathcal{F}([\varphi(X_1, \dots, X_p), \psi(X_{p+1}, \dots, X_{p+q})]).
\end{aligned}$$

Indeed, applying formulas (4.4) and (3.1) we easily get

$$\begin{aligned}
&[\mathcal{F}\varphi(\mathrm{af}(a_1) \circ \mathcal{F}X_1, \dots, \mathrm{af}(a_p) \circ \mathcal{F}X_p),\\
&\qquad \mathcal{F}\psi(\mathrm{af}(a_{p+1}) \circ \mathcal{F}X_{p+1}, \dots, \mathrm{af}(a_{p+q}) \circ \mathcal{F}X_{p+q})]\\
&\qquad = [\mathrm{af}(a_1 \cdot \ldots \cdot a_p) \circ \mathcal{F}(\varphi(X_1, \dots, X_p)),\\
&\qquad\quad \mathrm{af}(a_{p+1} \cdot \ldots \cdot a_{p+q}) \circ \mathcal{F}(\psi(X_{p+1}, \dots, X_{p+q}))]\\
&\qquad = \mathrm{af}(a_1 \cdot \ldots \cdot a_{p+q}) \circ \mathcal{F}([\varphi(X_1, \dots, X_p), \psi(X_{p+1}, \dots, X_{p+q})]).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathcal{F}\psi([\underline{\mathcal{F}\varphi}(\mathrm{af}(a_1) \circ \mathcal{F}X_1, \dots, \mathrm{af}(a_p) \circ \mathcal{F}X_p), \mathrm{af}(a_{p+1}) \circ \mathcal{F}X_{p+1}],\\
&\qquad \mathrm{af}(a_{p+2}) \circ \mathcal{F}X_{p+2}, \dots, \mathrm{af}(a_{p+q}) \circ \mathcal{F}X_{p+q})\\
&\qquad\quad = \mathrm{af}(a_1 \cdot \ldots \cdot a_{p+q}) \circ \mathcal{F}(\psi([\underline{\varphi}(X_1, \dots, X_p), X_{p+1}], X_{p+2}, \dots, X_{p+q}))
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{F}\psi(\underline{\mathcal{F}\varphi}([\mathrm{af}(a_1) \circ \mathcal{F}X_1, \mathrm{af}(a_2) \circ \mathcal{F}X_2], \mathrm{af}(a_3) \circ \mathcal{F}X_3, \dots, \mathrm{af}(a_{p+1}) \circ \mathcal{F}X_{p+1}),\\
&\qquad \mathrm{af}(a_{p+2}) \circ \mathcal{F}X_{p+2}, \dots, \mathrm{af}(a_{p+q}) \circ \mathcal{F}X_{p+q})\\
&\qquad\quad = \mathrm{af}(a_1 \cdot \ldots \cdot a_{p+q}) \circ \mathcal{F}(\psi(\underline{\varphi}([X_1, X_2], X_3, \dots, X_{p+1}), X_{p+2}, \dots, X_{p+q})),
\end{aligned}$$

and the same formulas with φ replaced by ψ and vice versa, and the same formulas with indices $1, \dots, p+q$ replaced by $\sigma(1), \dots, \sigma(p+q)$. Now, using the above formulas and formula (4.4) for $[[\varphi, \psi]]$ instead of φ and formula (2.2) on the Frolicher–Nijenhuis bracket $[[\varphi, \psi]]$ and formula (2.2) with φ and ψ replaced by $\mathcal{F}\varphi$ and $\mathcal{F}\psi$, and the $\mathbf{R}$-linearity of the complete lifting of vector fields (Lemma 3.3), we get

$$
\begin{aligned}
&\mathcal{F}([[\varphi,\psi]])(\mathrm{af}(a_1)\circ\mathcal{F}X_1,\dots,\mathrm{af}(a_{p+q})\circ\mathcal{F}X_{p+q})\\
&\quad=\mathrm{af}(a)\circ\mathcal{F}([[\varphi,\psi]](X_1,\dots,X_{p+q}))\\
&\quad=\frac{1}{p!q!}\sum_\sigma \operatorname{sgn}\sigma\cdot\mathrm{af}(a)\circ\mathcal{F}([\varphi(X_{\sigma 1},\dots,X_{\sigma p}),\psi(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})])\\
&\qquad+\frac{-1}{p!(q-1)!}\sum_\sigma \operatorname{sgn}\sigma\cdot\mathrm{af}(a)\circ\mathcal{F}(\psi([\underline{\varphi}(X_{\sigma 1},\dots,X_{\sigma p}),X_{\sigma(p+1)}],X_{\sigma(p+2)},\dots))\\
&\qquad+\frac{(-1)^{pq}}{(p-1)!q!}\sum_\sigma \operatorname{sgn}\sigma\cdot\mathrm{af}(a)\circ\mathcal{F}(\varphi([\underline{\psi}(X_{\sigma 1},\dots,X_{\sigma q}),X_{\sigma(q+1)}],X_{\sigma(q+2)},\dots))\\
&\qquad+\frac{(-1)^{p-1}}{(p-1)!(q-1)!2!}\sum_\sigma \operatorname{sgn}\sigma\cdot\mathrm{af}(a)\circ\mathcal{F}(\psi(\underline{\varphi}([X_{\sigma 1},X_{\sigma 2}],X_{\sigma 3},\dots),X_{\sigma(p+2)},\dots))\\
&\qquad+\frac{(-1)^{(p-1)q}}{(p-1)!(q-1)!2!}\sum_\sigma \operatorname{sgn}\sigma\cdot\mathrm{af}(a)\circ\mathcal{F}(\varphi(\underline{\psi}([X_{\sigma_1},X_{\sigma 2}],X_{\sigma 3},\dots),X_{\sigma(q+2)},\dots))\\
&\quad=\frac{1}{p!q!}\sum_\sigma \operatorname{sgn}\sigma\cdot[\mathcal{F}\varphi(\mathrm{af}(a_{\sigma 1})\circ\mathcal{F}X_{\sigma 1},\dots),\mathcal{F}\psi(\mathrm{af}(a_{\sigma(p+1)})\circ\mathcal{F}X_{\sigma(p+1)},\dots)]\\
&\qquad+\frac{-1}{p!(q-1)!}\sum_\sigma \operatorname{sgn}\sigma\cdot\mathcal{F}\psi([\underline{\mathcal{F}\varphi}(\mathrm{af}(a_{\sigma 1})\circ\mathcal{F}X_{\sigma 1},\dots),\mathrm{af}(a_{\sigma(p+1)})\circ\mathcal{F}X_{\sigma(p+1)}],\dots)\\
&\qquad+\frac{(-1)^{pq}}{(p-1)!q!}\sum_\sigma \operatorname{sgn}\sigma\cdot\mathcal{F}\varphi([\underline{\mathcal{F}\psi}(\mathrm{af}(a_{\sigma 1})\circ\mathcal{F}X_{\sigma 1},\dots),\mathrm{af}(a_{\sigma(p+1)})\circ\mathcal{F}X_{\sigma(p+1)}],\dots)\\
&\qquad+\frac{(-1)^{p-1}}{(p-1)!(q-1)!2!}\sum_\sigma \operatorname{sgn}\sigma\cdot\mathcal{F}\psi(\underline{\mathcal{F}\varphi}([\mathrm{af}(a_{\sigma 1})\circ\mathcal{F}X_{\sigma 1},\mathrm{af}(a_{\sigma 2})\circ\mathcal{F}X_{\sigma 2}],\dots),\dots)\\
&\qquad+\frac{(-1)^{(p-1)q}}{(p-1)!(q-1)!2!}\sum_\sigma \operatorname{sgn}\sigma\cdot\mathcal{F}\varphi(\underline{\mathcal{F}\psi}([\mathrm{af}(a_{\sigma 1})\circ\mathcal{F}X_{\sigma 1},\mathrm{af}(a_{\sigma 2})\circ\mathcal{F}X_{\sigma 2}],\dots),\dots)\\
&\quad=[[\mathcal{F}\varphi,\mathcal{F}\psi]](\mathrm{af}(a_1)\circ\mathcal{F}X_1,\dots,\mathrm{af}(a_{p+q})\circ\mathcal{F}X_{p+q}),
\end{aligned}
$$

for any vector fields $X_1,\dots,X_{p+q}$ on M and any $a_1,\dots,a_{p+q}\in A^F$, where $a:=a_1\cdot\ldots\cdot a_{p+q}$. Then, since the vector fields $\mathrm{af}(a)\circ\mathcal{F}X$ generate (over $C^\infty(FM)$) the space $\mathcal{X}(FM)$, formula (5.1) holds. □

6. An application to double-linear general connections

Let F be a ppgb-functor on $\mathcal{DVB}$.

In Definition 1.5, we introduced the concept of double-linear connections Γ in a double vector bundle K.

Lemma 6.1. *Given a double linear connection Γ in K, its complete lift $\mathcal{F}\Gamma$ is a double-linear connection in FK.*

Proof. Since $\Gamma(X)$ is a double-linear vector field on K with the underlying vector field equal to X, we have that $\mathcal{F}\Gamma(\mathrm{af}(a)\circ\mathcal{F}X)=\mathrm{af}(a)\cdot\mathcal{F}(\Gamma(X))$ is a double-linear vector field with the underlying vector field equal to $\mathrm{af}(a)\circ\mathcal{F}X$. Consequently, for any vector field $Y\in\mathcal{X}(FM)$, $\mathcal{F}\Gamma(Y)$ is a double linear vector field with the underlying vector field equal to Y. □

Definition 6.2. A *curvature* of a double linear connection Γ in a double vector bundle K is $\mathcal{R}_\Gamma := \frac{1}{2}[[\Gamma,\Gamma]] : K \to \wedge^2 T^*M \otimes VK$ (i.e., $\mathcal{R}_\Gamma(X,Y) = [\Gamma(X),\Gamma(Y)] - \Gamma([X,Y])$).

Theorem 6.3. *We have*

$$\mathcal{R}_{\mathcal{F}\Gamma} = \mathcal{F}(\mathcal{R}_\Gamma).$$

Proof. It is clear because of $\mathcal{F}([[\Gamma,\Gamma]]) = [[\mathcal{F}\Gamma,\mathcal{F}\Gamma]]$. □

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