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# A CANONICAL DISTRIBUTION ON ISOPARAMETRIC SUBMANIFOLDS II

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Abstract. The present paper continues our previous work [*Rev. Un. Mat. Argentina* **61** (2020), no. 1, 113–130], which was devoted to showing that on every compact, connected homogeneous isoparametric submanifold $M$ of codimension $h \geq 2$ in a Euclidean space, there exists a canonical distribution which is bracket generating of step 2. In that work this fact was established for the case when the system of restricted roots is reduced. Here we complete the proof of the main result for the case in which the system of restricted roots is $(BC)_q$, i.e., non-reduced.

## 1. Introduction

We present here the second part of the paper [4] devoted to indicating some properties of compact, connected *homogeneous* isoparametric submanifolds of Euclidean spaces of codimension $h \geq 2$. It is well known (see [5]) that all compact, connected, isoparametric submanifolds of Euclidean spaces of codimension $h \geq 3$ are homogeneous. On the other hand, in codimension $h = 2$ there are infinitely many non-homogeneous examples. Here we study the case of those spaces in which the restricted roots form the *non-reduced* system $\Phi(\mathfrak{g}_0, \mathfrak{a}_0) = (BC)_q$. This case was not included in [4], where only *reduced* systems of restricted roots were considered.

The compact, connected *homogeneous* isoparametric submanifolds of Euclidean spaces of codimension $h \geq 2$ (considered here and in [4]) are the principal orbits of the tangential representations of the compact, connected, irreducible, symmetric spaces. In Table 1, in the next section, the reader can find the symmetric spaces whose tangential principal orbits are the isoparametric submanifolds considered here. We shall use the notation and basic facts from [4, Sections 2–5] and, hoping that the reader has the opportunity to take a look at that work, we shall not repeat these facts here (except for necessary notation and formulae). Recall that the theorem to be proved is:

**Theorem 1.1.** *On any compact, connected, homogeneous isoparametric submanifold (for a real simple noncompact Lie algebra $\mathfrak{g}_0$) there exist a smooth* completely non-integrable *(i.e.,* bracket generating*)* step 2 *distribution $\mathfrak{D} \subset T(M^n)$, canonically associated to the manifold.*

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A distribution $\mathfrak{D}$ of $r$-planes ($n > r \geq 2$) in a connected manifold $M^n$ is *smooth* [6, p. 41] if for any $p \in M^n$ there is an open set $A$ containing $p$ and $r$ smooth vector fields $\{X_1, \ldots, X_r\}$ defined on $A$ such that $X_j(q) \in \mathfrak{D}(q)$ and $\mathfrak{D}(q) = \operatorname{span}_{\mathbb{R}}\{X_j(q)\}$, $1 \leq j \leq r$, for all $q \in A$. The distribution $\mathfrak{D}$ is said to be *completely non-integrable of* ***step 2*** if for every point $p \in M^n$ the above vector fields defined in $A$ satisfy, for all $q \in A$,

$$\operatorname{span}_{\mathbb{R}}\{X_j(q), [X_k, X_j](q) : 1 \leq k, j \leq r\} = T_q(M),$$

i.e., the generated real vector space *coincides* with the tangent space. The distribution $\mathfrak{D} = \mathfrak{D}(\Omega)$ mentioned in the theorem is defined in [4, Section 5]. In our present situation the system of restricted roots is $\Phi(\mathfrak{g}_0, \mathfrak{a}_0) = (BC)_q$ and the proof of Theorem 1.1 is naturally divided into *three* parts by the *nature* of the restricted roots.

This paper is organized as follows. In the next section we indicate, in Table 1, the symmetric spaces whose tangential representations contain the isoparametric submanifolds concerning us here. In Section 3 we recall the root system $(BC)_q$ (compare [2, p. 475, 3.25]) and in Section 4 we present the required four lemmata about the relation between the roots of $\mathfrak{g}_0^{\mathbb{C}}$ and their restricted counterpart $\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$. In Section 5 we recall required notation introduced in [4], and in Section 6 the formulae, from [4], needed in the proof of Theorem 1.1. Finally Section 7 contains the proof of Theorem 1.1 itself.

We include also an Appendix with proofs of the lemmata in Section 3, for the spaces in Table 1 but restricting the sizes of the root sistems $(BC)_q$ only to the case $q = 2$. This is intended to be *an example* of the way to obtain the required lemmata for the spaces in Table 1.

## 2. The spaces considered

In Table 1 we indicate the list of the symmetric spaces whose tangential principal orbits are the isoparametric submanifolds considered in the present paper. They are the spaces for which the corresponding systems of restricted roots are non-reduced, that is, $\Phi(\mathfrak{g}_0, \mathfrak{a}_0) = (BC)_n$ (for $n \geq 1$). We indicate only the compact spaces; they can, of course, be replaced by their corresponding non-compact duals.

## 3. The system $(BC)_q$

As mentioned above, we consider here the case in which $\Phi(\mathfrak{g}_0, \mathfrak{a}_0) = (BC)_q$. So we start recalling the system $(BC)_q$. We use the description in [2, p. 475, 3.25], for this *non-reduced* system of roots. The roots (written in terms of a set $\{\varepsilon_1, \ldots, \varepsilon_q\}$) are:

$$\begin{aligned} &\pm\varepsilon_i \pm \varepsilon_j, && 1 \leq i < j \leq q, \\ &\pm\varepsilon_i,\ \pm 2\varepsilon_i, && 1 \leq i \leq q. \end{aligned}$$

A system of simple roots $\Delta(\mathfrak{g}_0, \mathfrak{a}_0) = \{\lambda_1, \ldots, \lambda_q\}$ is defined by $\lambda_i = (\varepsilon_i - \varepsilon_{i+1})$ $(1 \leq i \leq q-1)$ and $\lambda_q = \varepsilon_q$. We write down all the *positive roots* in terms of the

Table 1.

| **non-reduced** | **space** | $\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$ |
|---|---|---|
| *AIII* | $SU(p+q)/S(U(p) \times U(q))$ $(p > q)$ | $(BC)_q$ |
| *AIV* | $SU(p+1)/S(U(p) \times U(1))$ $(p > 1)$ | $(BC)_1$ |
| *CII* | $Sp(p+q)/Sp(p) \times Sp(q)$ $(p > q)$ | $(BC)_q$ |
| *DIII* | $SO(2p)/U(p)$, $p > 1$, odd | $(BC)_{\frac{1}{2}(p-1)}$ |
| *EIII* | $E_6/(SO(10)T)$ | $(BC)_2$ |
| *FII* | $F_4/SO(9)$ | $(BC)_1$ |

simple ones. The double roots are

$$2\varepsilon_j = 2(\lambda_j + \ldots + \lambda_{q-1} + \lambda_q), \quad 1 \leq j \leq q,$$

and the others are

$$\begin{aligned} \varepsilon_j &= (\lambda_j + \ldots + \lambda_{q-1} + \lambda_q), & 1 \leq j \leq q, \\ \varepsilon_i - \varepsilon_j &= (\lambda_i + \ldots + \lambda_{j-1}), & 1 \leq i < j \leq q, \\ \varepsilon_i + \varepsilon_j &= (\lambda_i + \ldots + \lambda_{j-1}) + 2(\lambda_j + \ldots + \lambda_q), & 1 \leq i < j \leq q. \end{aligned} \tag{3.1}$$

## 4. Required lemmata

As in [4], for $\alpha$ in $\Phi(\mathfrak{g}, \mathfrak{h})$ or $\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$ we write $|\alpha| = \alpha$ if $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})$ (resp. $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$) and $|\alpha| = (-\alpha)$ if $(-\alpha) \in \Phi^+(\mathfrak{g}, \mathfrak{h})$ (resp. $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$).

Recall that $\Omega$ is the subset of $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ consisting of the roots of odd height, while $\Gamma = \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) - \Omega$.

**Lemma 4.1.** *Let us assume that $\Phi(\mathfrak{g}_0, \mathfrak{a}_0) = (BC)_q$. Given $\gamma \in \Gamma \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$* ***which is not a double root****, we can find $\eta \neq \delta$ in $\Omega \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ such that $\gamma = \eta + \delta$ and $|\eta - \delta|$ is not a root of $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$.*

*Proof.* We may assume that $q \geq 2$ because in $(BC)_1$, $\Gamma = \{2\lambda_1\}$. To prove the Lemma, we have to consider the roots in (3.1). Those in $\Gamma$ (written in terms of the simple ones) have an *even number* of coefficients equal to 1, while those in $\Omega$ have an odd number of them. Then $\gamma \in \Gamma$ must have at least *two coefficients* equal to 1. So we may suppress the simple root of lower index (call it $\eta$) and calling $\delta$ the sum of the remaining terms we have obviously $\gamma = \eta + \delta$ and $|\eta - \delta|$ is not a root of $(BC)_q$. This completes the proof of Lemma 4.1. □

The proofs of the following three lemmata are obtained by inspection in the pairs $\left(\Phi^+(\mathfrak{g}, \mathfrak{h}), (BC)_q\right)$. We include in the Appendix an *example* of the proof of Lemmata 4.2, 4.3, and 4.4.

**Lemma 4.2.** *Given $\lambda \in \Gamma \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) = (BC)_q$ **(which is not a double root)**, by Lemma 4.1 there exist two roots $\eta \neq \delta$ in $\Omega \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$, such that $\lambda = \eta + \delta$ and $|\eta - \delta|$ is not a root of $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$. Then **for any** root $\gamma \in \rho^{-1}(\lambda) \subset \Phi^+(\mathfrak{g}, \mathfrak{h})$ there exist roots $\alpha \in \rho^{-1}(\eta)$ and $\beta \in \rho^{-1}(\delta)$ such that $\gamma = \alpha + \beta$.*

Let us consider now the case in which $\lambda \in \Gamma \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) = (BC)_q$ *is a double root* ($q \geq 1$). Then we have two possibilities, which are considered separately in the following two lemmata.

**Lemma 4.3.** *Assume that $\lambda \in \Gamma$, $\lambda = 2\mu$, and $\mu \in \Omega$, and let $\gamma \in \rho^{-1}(\lambda)$. Then there exist roots $\alpha, \beta \in \rho^{-1}(\mu)$ such that $\alpha \neq \beta$ and $\gamma = \alpha + \beta$.*

**Lemma 4.4.** *Assume that $\lambda \in \Gamma$, $\lambda = 2\mu$, and $\mu \in \Gamma$. Since $\mu \in \Gamma$ is not a double root, by Lemma 4.1 there exist two roots $\eta \neq \delta$ in $\Omega \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$, such that $\mu = \eta + \delta$ and $|\eta - \delta|$ is not a root of $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) = (BC)_q$. Then* for any *root $\gamma \in \rho^{-1}(\lambda)$ there exist roots $\alpha_1 \neq \beta_1 \in \rho^{-1}(\eta)$ and $\alpha_2 \neq \beta_2 \in \rho^{-1}(\delta)$ such that $\alpha_1 + \alpha_2$ and $\beta_1 + \beta_2$ belong to $\rho^{-1}(\mu)$, and furthermore $\gamma = \alpha_1 + \alpha_2 + \beta_1 + \beta_2$.*

## 5. Basic notation

In order to avoid repetitions on the notation and basic facts we appeal to the patience of the reader and expect that he/she has the opportunity to take a look at [4, Sections 2–5], where the essential notation is introduced. We shall indicate the numbers of the formulae there in the corresponding references.

We start with the tangent space [4, Eq. (3.1)] at the basic point $E$ of our homogeneous isoparametric submanifold $M = Ad(K)E$ and recall the basis $\Xi_{\mathfrak{p}}(\lambda)$ of the subspaces $\mathfrak{p}_{0,\lambda}$, where $\lambda \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ [4, Eq. (4.8)]:

$$\Xi_{\mathfrak{p}}(\lambda) = \left\{ W_\varphi, U_\gamma, V_\gamma : \varphi \in \rho^{-1}(\lambda)_{\mathbb{R}}, \gamma \in \rho^{-1}(\lambda)^*_{\mathbb{C}} \right\}, \tag{5.1}$$

where $\rho^{-1}(\lambda)_{\mathbb{R}}$ is the set of *real roots* ($\alpha = \alpha^\sigma$) with image $\lambda$ by restriction ($\rho : \Phi(\mathfrak{g}, \mathfrak{h}) \to \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$) to $\mathfrak{a}_0$ and $\rho^{-1}(\lambda)^*_{\mathbb{C}}$ is the subset of the set of *complex roots* ($\alpha \neq \alpha^\sigma$) with image $\lambda$ by $\rho$, formed with one element of each pair $\{\alpha, \alpha^\sigma\}$ [4, §4.1]. The vectors $W_\alpha$ *shall be considered only for the real* roots $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ and they are determined by the equalities

$$\begin{aligned} &\text{if } k_\alpha = 1, && V_\alpha = 0,\ W_\alpha = U_\alpha, \\ &\text{if } k_\alpha = -1, && U_\alpha = 0,\ W_\alpha = V_\alpha, \end{aligned} \tag{5.2}$$

where the integers $k_\alpha$ are associated to the roots $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ and are defined in [4, Eqs. (4.1)–(4.3)].

Each of the tangent vectors at $E$ in $\Xi_{\mathfrak{p}}(\lambda)$ generates a corresponding local field $W_\alpha^F$, $U_\beta^F$, $V_\beta^F$ around $E$ as in [4, Eq. (5.1)]; for instance, for $U_\beta$, the local field $U_\beta^F$ is defined by

$$U_\beta^F(Ad(\exp tL)E) := Ad(\exp tL)U_\beta, \quad \text{for all } L \in S\left(0, \frac{r}{2}\right),\ t \in [0,1), \tag{5.3}$$

and similarly for $V_\beta$ and $W_\alpha$.

## 6. Known formulae

We need to recall also some of the formulae established in [4, Appendix]. For roots $\lambda,\ \mu \in \Omega \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$, $\delta \in \rho^{-1}(\lambda)$, and $\varphi \in \rho^{-1}(\mu)$ we have the identities

$$\begin{aligned} \Theta_{(\lambda,\mu,\delta,\varphi)} U_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)}\left(Ta\left(H_1\right)\right) &= \left[U_\delta^F, U_\varphi^F\right](E) - \left[V_\delta^F, V_\varphi^F\right](E), \\ \Theta_{(\lambda,\mu,\delta,\varphi)} V_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)}\left(Ta\left(T_2\right)\right) &= \left[U_\delta^F, V_\varphi^F\right](E) + \left[V_\delta^F, U_\varphi^F\right](E), \end{aligned} \tag{6.1}$$

$$H_1 = 2\left(k_\delta c_{\delta^\sigma,-\varphi}\left(x_{-\delta^\sigma+\varphi} - x_{\delta^\sigma-\varphi}\right) - k_\varphi c_{\delta,-\varphi^\sigma}\left(x_{\delta-\varphi^\sigma} - x_{-\delta+\varphi^\sigma}\right)\right), \tag{6.2}$$

$$T_2 = 2i\left(k_\delta c_{\delta^\sigma,-\varphi}\left(x_{\delta^\sigma-\varphi} + x_{-\delta^\sigma+\varphi}\right) - k_\varphi c_{\delta,-\varphi^\sigma}\left(x_{\delta-\varphi^\sigma} + x_{-\delta+\varphi^\sigma}\right)\right), \tag{6.3}$$

where $k_\alpha$ is defined in [4, Eqs. (4.1)–(4.3)] and the involved functions of $(\lambda, \mu, \delta, \varphi)$ and $(\lambda, \mu)$ are, respectively,

$$\begin{aligned} \Theta_{(\lambda,\mu,\delta,\varphi)} &= \left(2c_{\delta,\varphi}\right)\left(\frac{\lambda(E)+\mu(E)}{\lambda(E)\mu(E)}\right) \neq 0, \\ \Lambda_{(\lambda,\mu)} &= \left(\frac{\lambda(E)-\mu(E)}{\lambda(E)\mu(E)}\right). \end{aligned} \tag{6.4}$$

We have to add the case in which $(\delta+\varphi)$ is real and both $\delta$ and $\varphi$ are complex. Again $\lambda,\ \mu \in \Omega \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$, $\delta \in \rho^{-1}(\lambda)^*_{\mathbb{C}}$, and $\varphi \in \rho^{-1}(\mu)^*_{\mathbb{C}}$. In this case, from (6.1) and having (5.2) in mind we have

$$\begin{aligned} & k_{(\delta+\varphi)} = 1, \\ & \Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)}\left(Ta\left(H_1\right)\right) = \left[U_\delta^F, U_\varphi^F\right](E) - \left[V_\delta^F, V_\varphi^F\right](E), \\ & k_{(\delta+\varphi)} = (-1), \\ & \Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)}\left(Ta\left(T_2\right)\right) = \left[U_\delta^F, V_\varphi^F\right](E) + \left[V_\delta^F, U_\varphi^F\right](E). \end{aligned} \tag{6.5}$$

It is important to notice that when we have

$$\textbf{either} \quad H_1 = 0 = T_2 \quad \textbf{or} \quad \Lambda_{(\lambda,\mu)} = 0 \tag{6.6}$$

then (6.5) reduces to

$$\Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} = \begin{cases} \left[U_\delta^F, U_\varphi^F\right](E) - \left[V_\delta^F, V_\varphi^F\right](E), \\ \left[U_\delta^F, V_\varphi^F\right](E) + \left[V_\delta^F, U_\varphi^F\right](E). \end{cases} \tag{6.7}$$

We need also the case in which both $\delta$ and $\varphi$ are real. That is $\lambda,\ \mu \in \Omega \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$, $\delta \in \rho^{-1}(\lambda)_{\mathbb{R}}$, and $\varphi \in \rho^{-1}(\mu)_{\mathbb{R}}$. We must recall also that in the present case [4, Eq. (8.1)] yields $k_\delta k_\varphi = k_{(\delta+\varphi)}$ and formulae (6.5) become

$$\begin{aligned} & k_{(\delta+\varphi)} = 1,\ k_\delta = k_\varphi = 1, \\ & \Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)} \left(Ta\left(H_1\right)\right) = \left[W_\delta^F, W_\varphi^F\right](E), \\ & k_{(\delta+\varphi)} = 1,\ k_\delta = k_\varphi = -1, \\ & \Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)} \left(Ta\left(H_1\right)\right) = -\left[W_\delta^F, W_\varphi^F\right](E), \\ & k_{(\delta+\varphi)} = (-1),\ k_\delta = 1,\ k_\varphi = -1, \\ & \Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)} \left(Ta\left(T_2\right)\right) = \left[W_\delta^F, W_\varphi^F\right](E), \\ & k_{(\delta+\varphi)} = (-1),\ k_\delta = -1,\ k_\varphi = 1, \\ & \Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)} \left(Ta\left(T_2\right)\right) = \left[W_\delta^F, W_\varphi^F\right](E). \end{aligned} \tag{6.8}$$

Here again we have that if (6.6) holds then (6.8) can be reduced to

$$\Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} = \pm\left[W_\delta^F, W_\varphi^F\right](E). \tag{6.9}$$

## 7. Proof of Theorem 1.1

Let us take $\lambda \in \Gamma$ and recall the basis (5.1) of $\mathfrak{p}_{0\lambda}$.

**Remark 7.1.** It is important to observe that in order to prove Theorem 1.1, it is enough to show (*for each* $\lambda \in \Gamma$) that the vectors of the basis $\Xi_{\mathfrak{p}}(\lambda)$ of $\mathfrak{p}_{0\lambda} \subset T_E(M)$ may be computed as a sum of brackets of local fields (defined around $E$) that belong to the distribution $\mathfrak{D}(\Omega)$. So this is the objective here.

To that end, we divide our considerations into three cases which require somewhat different procedures, namely

(A) $\lambda$ is not a double root,
(B) $\lambda = 2\mu \in \Gamma$ and $\mu \in \Omega$,
(C) $\lambda = 2\mu \in \Gamma$ and $\mu \in \Gamma$.

7.1. **Case (A).** Let us take $\lambda \in \Gamma$ (*which is not a double root*) and consider the basis (5.1) of $\mathfrak{p}_{0\lambda}$. For this $\lambda \in \Gamma$, take $\gamma \in \left(\rho^{-1}(\lambda)_{\mathbb{C}}^*\right)$ and consider $U_\gamma$, $V_\gamma$ for our chosen $\gamma$. By Lemma 4.1, there exist roots $\eta$ and $\delta$ in $\Omega \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ such that

$$\eta \neq \delta, \quad \lambda = \eta + \delta, \quad |\eta - \delta| \text{ is not a root of } \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0). \tag{7.1}$$

Furthermore, by Lemma 4.2, for the root $\gamma \in \rho^{-1}(\lambda)$ there exist roots $\alpha \in \rho^{-1}(\eta)$ and $\beta \in \rho^{-1}(\delta)$ such that $\gamma = \alpha + \beta$. Then we have $U_\gamma = U_{(\alpha+\beta)}$, $V_\gamma = V_{(\alpha+\beta)}$ and so formulae (6.1) (replacing the actual roots) are:

$$\begin{aligned} \Theta_{(\eta,\delta,\alpha,\beta)} U_{(\alpha+\beta)} + \Lambda_{(\eta,\delta)} Ta\left(H_1\right) &= \left[U_\alpha^F, U_\beta^F\right](E) - \left[V_\alpha^F, V_\beta^F\right](E), \\ \Theta_{(\eta,\delta,\alpha,\beta)} V_{(\alpha+\beta)} + \Lambda_{(\eta,\delta)} Ta\left(T_2\right) &= \left[U_\alpha^F, V_\beta^F\right](E) + \left[V_\alpha^F, U_\beta^F\right](E). \end{aligned} \tag{7.2}$$

Let us study now the terms $H_1$ and $T_2$ (see (6.2) and (6.3)), for the pair of roots $(\alpha, \beta)$. They are:

$$\begin{aligned} H_1 &= 2k_\alpha c_{\alpha^\sigma,-\beta}\left(x_{-\alpha^\sigma+\beta} - x_{\alpha^\sigma-\beta}\right) - 2k_\beta c_{\alpha,-\beta^\sigma}\left(x_{\alpha-\beta^\sigma} - x_{-\alpha+\beta^\sigma}\right), \\ T_2 &= 2ik_\alpha c_{\alpha^\sigma,-\beta}\left(x_{\alpha^\sigma-\beta} + x_{-\alpha^\sigma+\beta}\right) - 2ik_\beta c_{\alpha,-\beta^\sigma}\left(x_{\alpha-\beta^\sigma} + x_{-\alpha+\beta^\sigma}\right). \end{aligned}$$

In our present situation we have

$$|\beta-\alpha^\sigma| \text{ and } |\alpha-\beta^\sigma| \text{ are not roots of } \Phi(\mathfrak{g},\mathfrak{h}). \tag{7.3}$$

In fact, since $\eta$ and $\delta$ satisfy (7.1) and $\alpha \in \rho^{-1}(\eta)$, $\beta \in \rho^{-1}(\delta)$, if $|\beta-\alpha^\sigma|$ were a root of $\Phi(\mathfrak{g},\mathfrak{h})$ then $\rho(|\beta-\alpha^\sigma|) = |\eta-\delta|$ would be a root of $\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)$, which is not the case by (7.1). Similarly, $|\alpha-\beta^\sigma|$ is not a root of $\Phi(\mathfrak{g},\mathfrak{h})$. This clearly yields $H_1 = T_2 = 0$ and going back to (7.2) we see that $U_\gamma = U_{(\alpha+\beta)}$, $V_\gamma = V_{(\alpha+\beta)}$ are *sums of brackets (evaluated on $E$) of local fields defined around $E$ that belong to the distribution* $\mathfrak{D}(\Omega)$.

It remains to consider the case of $\varphi \in \left(\rho^{-1}(\lambda)_{\mathbb{R}}\right)$ for our taken $\lambda \in \Gamma \subset \Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)$. We have the vector $W_\varphi$ and again there exist two roots $\eta$ and $\delta$ in $\Omega \subset \Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)$ satisfying (7.1) and roots $\xi \in \rho^{-1}(\eta)$, $\omega \in \rho^{-1}(\delta)$ such that $\varphi = \xi+\omega$. Then we have the following possibilities:

$$\begin{aligned}&\text{(i) } \xi \text{ and } \omega \text{ are both real roots of } \Phi(\mathfrak{g},\mathfrak{h});\\ &\text{(ii) } \xi \text{ and } \omega \text{ are both complex roots of } \Phi(\mathfrak{g},\mathfrak{h}).\end{aligned} \tag{7.4}$$

(In the case (ii), $\varphi = \xi^\sigma + \omega^\sigma$ is another decomposition of $\varphi$).

Let us consider the case (i) in (7.4). We see, by the argument above, that (7.3) holds in this case and it takes the form

$$(\xi-\omega) \text{ and } (\omega-\xi) \text{ are not roots of } \Phi(\mathfrak{g},\mathfrak{h}).$$

Now, we see (*for the roots* $(\xi,\omega)$) that formulae (6.2) and (6.3) yield $H_1 = T_2 = 0$, and then (6.6) holds. Then we may write $W_\varphi = W_{(\xi+\omega)}$ using formula (6.9) (*for the roots* $(\xi,\omega)$) and therefore the vector $W_\varphi$ is a *bracket (evaluated at $E$) of local fields defined around $E$ that belong to the distribution* $\mathfrak{D}(\Omega)$.

On the other hand, in the case (ii) of (7.4) we have that (7.3) holds for the pair of complex roots $(\xi,\omega)$, which again yields $H_1 = T_2 = 0$. Then by formulae (6.7) (*for* $\varphi = \xi+\omega$) we have that also in this case $W_\varphi$ is a *sum of brackets (evaluated at $E$) of local fields that belong to the distribution* $\mathfrak{D}(\Omega)$. This completes the proof of Theorem 1.1 in Case (A).

7.2. **Case (B).** Let us take $\lambda \in \Gamma$ a double root, $\lambda = 2\mu$ with $\mu \in \Omega$, and consider again the basis (5.1) for $\mathfrak{p}_{0\lambda}$. As above we have to consider the two situations $\varphi \in \rho^{-1}(\lambda)_{\mathbb{R}}$ and $\gamma \in \rho^{-1}(\lambda)^*_{\mathbb{C}}$.

We shall take first $\varphi \in \rho^{-1}(\lambda)_{\mathbb{R}}$. Then we have to consider the vector $W_\varphi$. By Lemma 4.3 there exist roots $\alpha$, $\beta \in \rho^{-1}(\mu)$ such that $\alpha \neq \beta$ and $\varphi = \alpha+\beta$, and we have then (*for $\alpha$ and $\beta$*) the alternative (7.4), that is, either $\alpha$ and $\beta$ are both real or they are both complex roots.

Let us assume that *$\alpha$ and $\beta$ are both real.* Then since $\alpha$, $\beta \in \rho^{-1}(\mu)$, recalling (6.4) we have $\lambda_{(\mu,\mu)} = 0$, which means that (6.6) holds and then we may again use (6.9) (*for $(\alpha,\beta)$ instead of $(\delta,\varphi)$*) to write $W_\varphi$ as a *bracket (evaluated at $E$) of local fields defined around $E$ and belonging to the distribution* $\mathfrak{D}(\Omega)$.

Let us assume now that *$\alpha$ and $\beta$ are both complex roots.* Since $\alpha$, $\beta \in \rho^{-1}(\mu)$ $(\alpha \neq \beta)$ we again have, by (6.4), that $\Lambda_{(\mu,\mu)} = 0$, which means that (6.6) holds in this situation too and we may use formulae (6.7) (*with $(\alpha,\beta)$ instead of $(\delta,\varphi)$*) and also here the vector $W_\varphi$ is a sum of *brackets (evaluated at E) of local fields defined around E and belonging to the distribution* $\mathfrak{D}(\Omega)$.

Continuing in Case (B), we consider now the possibility $\gamma \in \rho^{-1}(\lambda)^*_{\mathbb{C}}$. By Lemma 4.3, there exist roots $\alpha, \beta \in \rho^{-1}(\mu)$ such that $\alpha \neq \beta$ and $\gamma = \alpha + \beta$. We have to look now at $U_\gamma = U_{(\alpha+\beta)}$, $V_\gamma = V_{(\alpha+\beta)}$. Recalling (6.1), since $\alpha, \beta \in \rho^{-1}(\mu)$, we have again $\Lambda_{(\mu,\mu)} = 0$ and once more (6.6) holds. Then (6.1) takes the form

$$\Theta_{(\mu,\mu,\alpha,\beta)}U_{(\alpha+\beta)} = \left[U^F_\alpha, U^F_\beta\right](E) - \left[V^F_\alpha, V^F_\beta\right](E),$$
$$\Theta_{(\mu,\mu,\alpha,\beta)}V_{(\alpha+\beta)} = \left[U^F_\alpha, V^F_\beta\right](E) + \left[V^F_\alpha, U^F_\beta\right](E),$$

and therefore the vectors $U_\gamma = U_{(\alpha+\beta)}$, $V_\gamma = V_{(\alpha+\beta)}$ are *sums of brackets (evaluated at E) of local fields defined around E belonging to the distribution* $\mathfrak{D}(\Omega)$. This completes the proof of Theorem 1.1 in Case (B).

7.3. **Case (C).** Here $\lambda \in \Gamma$ is a *double root*, $\lambda = 2\mu$ and $\mu \in \Gamma$. Since $\mu \in \Gamma$ *is not a double root*, by Lemma 4.1 there exist two roots $\eta \neq \delta$ in $\Omega$ such that $\mu = \eta + \delta$ and $|\eta - \delta|$ is not a root of $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$. Also, by Lemma 4.4, for the root $\gamma \in \rho^{-1}(\lambda)$, there exist roots $\alpha_1 \neq \beta_1 \in \rho^{-1}(\eta)$ and $\alpha_2 \neq \beta_2 \in \rho^{-1}(\delta)$ such that $\alpha_1 + \alpha_2$ and $\beta_1 + \beta_2$ belong to $\rho^{-1}(\mu)$ and $\gamma = \alpha_1 + \alpha_2 + \beta_1 + \beta_2$. In principle we may have that either $\gamma \in \rho^{-1}(\lambda)_{\mathbb{R}}$ or $\gamma \in \rho^{-1}(\lambda)^*_{\mathbb{C}}$. In order to simplify our notation we set

$$\alpha_{12} = (\alpha_1 + \alpha_2) \in \rho^{-1}(\mu), \quad \beta_{12} = (\beta_1 + \beta_2) \in \rho^{-1}(\mu).$$

We have to study again the vectors of the basis (5.1) for our $\lambda \in \Gamma$. But before doing this, we are going take a look at $U_{\alpha_{12}}$, $V_{\alpha_{12}}$, $U_{\beta_{12}}$, and $V_{\beta_{12}}$. For the roots involved, we have the following possibilities:

(a) $\alpha_{12}, \beta_{12} \in \rho^{-1}(\mu)^*_{\mathbb{C}}$, $\quad \alpha_1, \beta_1 \in \rho^{-1}(\eta)^*_{\mathbb{C}}$, $\quad \alpha_2, \beta_2 \in \rho^{-1}(\delta)^*_{\mathbb{C}}$;

(b) $\alpha_{12}, \beta_{12} \in \rho^{-1}(\mu)_{\mathbb{R}}$, $\quad \alpha_1, \beta_1 \in \rho^{-1}(\eta)_{\mathbb{R}}$, $\quad \alpha_2, \beta_2 \in \rho^{-1}(\delta)_{\mathbb{R}}$;

(c) $\alpha_{12}, \beta_{12} \in \rho^{-1}(\mu)_{\mathbb{R}}$, $\quad \alpha_1, \beta_1 \in \rho^{-1}(\eta)^*_{\mathbb{C}}$, $\quad \alpha_2, \beta_2 \in \rho^{-1}(\delta)^*_{\mathbb{C}}$.

Let us start taking $\gamma \in \rho^{-1}(2\mu)^*_{\mathbb{C}}$; then we are in situation (a) and considering again the basis (5.1), we have to study the vectors $U_\gamma$ and $V_\gamma$. But first we take a look at the vectors $U_{\alpha_{12}}$, $V_{\alpha_{12}}$, $U_{\beta_{12}}$, and $V_{\beta_{12}}$.

7.3.1. *Situation (a).* By (6.1) (*with* $(\alpha_1, \alpha_2)$ *instead of* $(\delta, \varphi)$) we have

$$\begin{aligned} \Theta_{(\eta,\delta,\alpha_1,\alpha_2)}U_{\alpha_{12}} + \Lambda_{(\eta,\delta)}Ta(H_1) &= \left[U^F_{\alpha_1}, U^F_{\alpha_2}\right](E) - \left[V^F_{\alpha_1}, V^F_{\alpha_2}\right](E), \\ \Theta_{(\eta,\delta,\alpha_1,\alpha_2)}V_{\alpha_{12}} + \Lambda_{(\eta,\delta)}Ta(T_2) &= \left[U^F_{\alpha_1}, V^F_{\alpha_2}\right](E) + \left[V^F_{\alpha_1}, U^F_{\alpha_2}\right](E). \end{aligned} \tag{7.5}$$

The extra terms $H_1$ and $T_2$ have expressions similar to (6.2) and (6.3) (*with* $(\alpha_1, \alpha_2)$ *instead of* $(\delta, \varphi)$). That is,

$$\begin{aligned} H_1 &= 2k_{\alpha_1}c_{\alpha_1^\sigma,-\alpha_2}\left(x_{-\alpha_1^\sigma+\alpha_2} - x_{\alpha_1^\sigma-\alpha_2}\right) - 2k_{\alpha_2}c_{\alpha_1,-\alpha_2^\sigma}\left(x_{\alpha_1-\alpha_2^\sigma} - x_{-\alpha_1+\alpha_2^\sigma}\right), \\ T_2 &= 2ik_{\alpha_1}c_{\alpha_1^\sigma,-\alpha_2}\left(x_{\alpha_1^\sigma-\alpha_2} + x_{-\alpha_1^\sigma+\alpha_2}\right) - 2ik_{\alpha_2}c_{\alpha_1,-\alpha_2^\sigma}\left(x_{\alpha_1-\alpha_2^\sigma} + x_{-\alpha_1+\alpha_2^\sigma}\right). \end{aligned} \tag{7.6}$$

For $U_{\beta_{12}}$ and $V_{\beta_{12}}$ we have similar expressions (with $\beta$ instead of $\alpha$).

Since $\alpha_1,\ \beta_1 \in \rho^{-1}(\eta)$, $\alpha_2,\ \beta_2 \in \rho^{-1}(\delta)$ (with $\lambda = \eta + \delta$ and $|\eta - \delta|$ not a root of $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$) we see, in the same way as above in (7.3), that

$$|\alpha_2 - \alpha_1^\sigma|\,,\ |\alpha_1 - \alpha_2^\sigma|\,,\ |\beta_2 - \beta_1^\sigma|\,,\ \text{and}\ |\beta_1 - \beta_2^\sigma|\ \text{are not roots of}\ \Phi(\mathfrak{g}, \mathfrak{h})\,, \tag{7.7}$$

and, once again, this yields $H_1 = T_2 = 0$ (for $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$). So the extra terms in (7.5) vanish.

Since the roots $\eta, \delta, \alpha_1, \alpha_2, \beta_1, \beta_2$ are fixed, in order to simplify notation we may set

$$a = \Theta_{(\eta,\delta,\alpha_1,\alpha_2)} \neq 0, \qquad b = \Theta_{(\eta,\delta,\beta_1,\beta_2)} \neq 0, \tag{7.8}$$

and hence, for $U_{\alpha_{12}}$, $V_{\alpha_{12}}$, $U_{\beta_{12}}$ and $V_{\beta_{12}}$, (7.5) yields

$$\begin{aligned} aU_{\alpha_{12}} &= \left[U^F_{\alpha_1}, U^F_{\alpha_2}\right](E) - \left[V^F_{\alpha_1}, V^F_{\alpha_2}\right](E),\\ aV_{\alpha_{12}} &= \left[U^F_{\alpha_1}, V^F_{\alpha_2}\right](E) + \left[V^F_{\alpha_1}, U^F_{\alpha_2}\right](E),\\ bU_{\beta_{12}} &= \left[U^F_{\beta_1}, U^F_{\beta_2}\right](E) - \left[V^F_{\beta_1}, V^F_{\beta_2}\right](E),\\ bV_{\beta_{12}} &= \left[U^F_{\beta_1}, V^F_{\beta_2}\right](E) + \left[V^F_{\beta_1}, U^F_{\beta_2}\right](E). \end{aligned} \tag{7.9}$$

Now we have to make an important remark.

**Remark 7.2.** For a diffeomorphism $\varphi$ between two open sets in a manifold, one has the definition of **$\varphi$-related smooth fields** (see for instance [6, p. 41, 1.54]). By the definition (5.3) used to generate the local fields $U^F_{\alpha_1}$, we see that

$$U^F_{\alpha_1}\left(Ad\left(\exp tL\right)(E)\right) = Ad\left(\exp tL\right)\left(U^F_{\alpha_1}(E)\right)$$

(and similarly for $U^F_{\alpha_2}, V^F_{\alpha_1}$ and $V^F_{\alpha_2}$). Then, since the brackets of $\varphi$-related smooth fields are $\varphi$-related (see [6, p. 41, 1.55]), we may write

$$\begin{aligned} \left[U^F_{\alpha_1}, U^F_{\alpha_2}\right]\left(Ad\left(\exp tL\right)(E)\right) &= Ad\left(\exp tL\right)\left(\left[U^F_{\alpha_1}, U^F_{\alpha_2}\right](E)\right),\\ \left[V^F_{\alpha_1}, V^F_{\alpha_2}\right]\left(Ad\left(\exp tL\right)(E)\right) &= Ad\left(\exp tL\right)\left(\left[V^F_{\alpha_1}, V^F_{\alpha_2}\right](E)\right). \end{aligned}$$

Then extending *locally* around $E$ the vector $U_{\alpha_{12}}$ as in (5.3) (with $\delta = \alpha_{12}$) to the local field $U^F_{\alpha_{12}}$, that is,

$$U^F_{\alpha_{12}}\left(Ad\left(\exp tL\right)E\right) := Ad\left(\exp tL\right)U_{\alpha_{12}}, \quad \text{for all } L \in S\left(0, \frac{r}{2}\right),\ t \in [0, 1),$$

and by defining similarly, locally around $E$, the fields $V^F_{\alpha_{12}}$, $U^F_{\beta_{12}}$, and $V^F_{\beta_{12}}$, we see (by [6, p. 41, 1.55]) that we may *extend* the four lines in (7.9) to *local equalities of fields around* $E$ and write

$$\begin{aligned} aU^F_{\alpha_{12}} &= \left[U^F_{\alpha_1}, U^F_{\alpha_2}\right] - \left[V^F_{\alpha_1}, V^F_{\alpha_2}\right]\\ aV^F_{\alpha_{12}} &= \left[U^F_{\alpha_1}, V^F_{\alpha_2}\right] + \left[V^F_{\alpha_1}, U^F_{\alpha_2}\right]\\ bU^F_{\beta_{12}} &= \left[U^F_{\beta_1}, U^F_{\beta_2}\right] - \left[V^F_{\beta_1}, V^F_{\beta_2}\right]\\ bV^F_{\beta_{12}} &= \left[U^F_{\beta_1}, V^F_{\beta_2}\right] + \left[V^F_{\beta_1}, U^F_{\beta_2}\right], \end{aligned} \tag{7.10}$$

where the fields on the left side and the brackets on the right side are evaluated at the same point in a neighborhood of $E$.

Now we go back to our root $\gamma \in \rho^{-1}(2\mu)^*_{\mathbb{C}}$. We take $U_\gamma$ and $V_\gamma$ members of the basis (5.1). Since $\gamma = \alpha_{12} + \beta_{12}$ ($\alpha_{12},\ \beta_{12} \in \rho^{-1}(\mu)$) we have for $U_\gamma = U_{(\alpha_{12}+\beta_{12})}$

and $V_\gamma = V_{(\alpha_{12}+\beta_{12})}$ expressions such as (6.1) with (6.2) and (6.3) with the roots $\alpha_{12}$ and $\beta_{12}$. Now, for convenience, we introduce the notation

$$\widehat{\Theta} = \Theta_{(\mu,\mu,\alpha_{12},\beta_{12})} \neq 0$$

and observe that in the equalities (6.1) we have $\Lambda_{(\lambda,\mu)}$ (defined in (6.4)) which in this situation is $\Lambda_{(\mu,\mu)} = 0$.

Then we see that in our present situation the equalities (6.1) become

$$\begin{aligned} \widehat{\Theta} U_\gamma &= \left[U^F_{\alpha_{12}}, U^F_{\beta_{12}}\right](E) - \left[V^F_{\alpha_{12}}, V^F_{\beta_{12}}\right](E) \\ \widehat{\Theta} V_\gamma &= \left[U^F_{\alpha_{12}}, V^F_{\beta_{12}}\right](E) + \left[V^F_{\alpha_{12}}, U^F_{\beta_{12}}\right](E). \end{aligned} \tag{7.11}$$

Now we may replace the fields inside the brackets on the right side of (7.11) by their expressions in (7.10). By doing this for $\widehat{\Theta} U_\gamma$ in (7.11) we see that the bracket $\left[U^F_{\alpha_{12}}, U^F_{\beta_{12}}\right]$ in the *first* term (using (7.10) and multiplying by $ab \neq 0$) is

$$\begin{aligned} ab\left[U^F_{\alpha_{12}}, U^F_{\beta_{12}}\right] &= \left[aU^F_{\alpha_{12}}, bU^F_{\beta_{12}}\right] \\ &= +\left[\left[U^F_{\alpha_1}, U^F_{\alpha_2}\right], \left[U^F_{\beta_1}, U^F_{\beta_2}\right]\right] - \left[\left[U^F_{\alpha_1}, U^F_{\alpha_2}\right], \left[V^F_{\beta_1}, V^F_{\beta_2}\right]\right] \\ &\quad - \left[\left[V^F_{\alpha_1}, V^F_{\alpha_2}\right], \left[U^F_{\beta_1}, U^F_{\beta_2}\right]\right] + \left[\left[V^F_{\alpha_1}, V^F_{\alpha_2}\right], \left[V^F_{\beta_1}, V^F_{\beta_2}\right]\right]. \end{aligned} \tag{7.12}$$

By proceeding similarly with the *second* term $\left[V^F_{\alpha_{12}}, V^F_{\beta_{12}}\right]$ in the first line of (7.11) we have

$$\begin{aligned} ab\left[V^F_{\alpha_{12}}, V^F_{\beta_{12}}\right] &= \left[aV^F_{\alpha_{12}}, bV^F_{\beta_{12}}\right] \\ &= \left[\left[U^F_{\alpha_1}, V^F_{\alpha_2}\right], \left[U^F_{\beta_1}, V^F_{\beta_2}\right]\right] + \left[\left[U^F_{\alpha_1}, V^F_{\alpha_2}\right], \left[V^F_{\beta_1}, U^F_{\beta_2}\right]\right] \\ &\quad + \left[\left[V^F_{\alpha_1}, U^F_{\alpha_2}\right], \left[U^F_{\beta_1}, V^F_{\beta_2}\right]\right] + \left[\left[V^F_{\alpha_1}, U^F_{\alpha_2}\right], \left[V^F_{\beta_1}, U^F_{\beta_2}\right]\right]. \end{aligned} \tag{7.13}$$

Now *by expanding* the brackets in each of the four terms in (7.12) and (7.13) and computing the difference of the expanded expressions (dividing by $ab$) we finally obtain the first line of (7.11). That is,

$$\begin{aligned} \left(\frac{\widehat{\Theta}}{ab}\right) U_\gamma = &-2\left[U^F_{\alpha_1}, V^F_{\beta_1}\right] - 2\left[U^F_{\alpha_2}, V^F_{\beta_1}\right] - 2\left[U^F_{\alpha_1}, V^F_{\beta_2}\right] - 2\left[U^F_{\alpha_2}, V^F_{\beta_2}\right] \\ &- 2\left[V^F_{\alpha_1}, U^F_{\beta_1}\right] - 2\left[V^F_{\alpha_2}, U^F_{\beta_1}\right] - 2\left[V^F_{\alpha_1}, U^F_{\beta_2}\right] - 2\left[V^F_{\alpha_2}, U^F_{\beta_2}\right], \end{aligned}$$

*where all brackets should be evaluated at $E$.*

By computing similarly (using again (7.10)), the second identity in (7.11) turns out to be

$$\begin{aligned} \left(\frac{\widehat{\Theta}}{ab}\right) V_\gamma = &\, 2\left[U^F_{\alpha_1}, U^F_{\beta_1}\right] + 2\left[U^F_{\alpha_2}, U^F_{\beta_1}\right] + 2\left[U^F_{\alpha_2}, U^F_{\beta_2}\right] + 2\left[U^F_{\alpha_1}, U^F_{\beta_2}\right] \\ &- 2\left[V^F_{\alpha_1}, V^F_{\beta_2}\right] - 2\left[V^F_{\alpha_2}, V^F_{\beta_1}\right] - 2\left[V^F_{\alpha_2}, V^F_{\beta_2}\right] - 2\left[V^F_{\alpha_1}, V^F_{\beta_1}\right], \end{aligned}$$

*where all the brackets should be evaluated at $E$.* Then we see that $\left(\frac{\widehat{\Theta}}{ab}\right) U_\gamma$ and $\left(\frac{\widehat{\Theta}}{ab}\right) V_\gamma$ *are linear combinations of brackets (evaluated at $E$) of fields in the distribution.* These formulae are valid for situation (a) and $\gamma \in \rho^{-1}(\lambda)^*_{\mathbb{C}}$.

Then we have the proof of the theorem in the situation (a).

Let us take now $\gamma \in \rho^{-1}(2\mu)_{\mathbb{R}}$; then we have to study situations (b) and (c).

7.3.2. *Situation (b).* We have

(b) $\alpha_{12}, \beta_{12} \in \rho^{-1}(\mu)_{\mathbb{R}}, \quad \alpha_1, \beta_1 \in \rho^{-1}(\eta)_{\mathbb{R}}, \quad \alpha_2, \beta_2 \in \rho^{-1}(\delta)_{\mathbb{R}}$.

Here we start by considering $\alpha_{12} \in \rho^{-1}(\mu)_{\mathbb{R}}$ with $\alpha_1 \in \rho^{-1}(\eta)_{\mathbb{R}}$ and $\alpha_2 \in \rho^{-1}(\delta)_{\mathbb{R}}$, and we have that

$$|\alpha_2 - \alpha_1^\sigma|, \ |\alpha_1 - \alpha_2^\sigma|, \ |\beta_2 - \beta_1^\sigma|, \text{ and } |\beta_1 - \beta_2^\sigma| \text{ are not roots of } \Phi(\mathfrak{g}, \mathfrak{h}).$$

By (7.6) and (7.7) (since $|\alpha_1 - \alpha_2|$ and $|\beta_2 - \beta_1^\sigma|$ are not roots) we have that this yields $H_1 = T_2 = 0$ (for $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$) and hence we may use (6.9), which for the present roots takes the form

$$\Theta_{(\eta,\delta,\alpha_1,\alpha_2)} W_{(\alpha_{12})} = \pm \left[W_{\alpha_1}^F, W_{\alpha_2}^F\right](E).$$

This identity extends locally as we did to get (7.10) and recalling (7.8) we may write

$$aW_{(\alpha_{12})}^F = \pm \left[W_{\alpha_1}^F, W_{\alpha_2}^F\right], \tag{7.14}$$

where both sides are evaluated at the same point around $E$.

Proceeding similarly for $\beta_{12} \in \rho^{-1}(\mu)_{\mathbb{R}}$, $\beta_1 \in \rho^{-1}(\eta)_{\mathbb{R}}$, and $\beta_2 \in \rho^{-1}(\delta)_{\mathbb{R}}$ we have

$$bW_{(\beta_{12})}^F = \pm \left[W_{\beta_1}^F, W_{\beta_2}^F\right]. \tag{7.15}$$

Now consider $\gamma = (\alpha_{12} + \beta_{12}) \in \rho^{-1}(2\mu)_{\mathbb{R}}$, with $\alpha_{12} \in \rho^{-1}(\mu)_{\mathbb{R}}$ and $\beta_{12} \in \rho^{-1}(\mu)_{\mathbb{R}}$. Here we have as above $\lambda_{(\mu,\mu)} = 0$ and (6.6) lets us use (6.9) again and write

$$\Theta_{(\mu,\mu,\alpha_{12},\beta_{12})} W_{((\alpha_{12}+\beta_{12}))} = \pm \left[W_{\alpha_{12}}^F, W_{\beta_{12}}^F\right](E).$$

Now setting

$$A = \frac{\Theta_{(\mu,\mu,\alpha_{12},\beta_{12})}}{ab}, \tag{7.16}$$

by replacing (7.14) and (7.15) and expanding we see that

$$AW_{((\alpha_{12}+\beta_{12}))} = \pm \left(\left[W_{\alpha_1}^F, W_{\beta_1}^F\right] + \left[W_{\alpha_1}^F, W_{\beta_2}^F\right] + \left[W_{\alpha_2}^F, W_{\beta_1}^F\right] + \left[W_{\alpha_2}^F, W_{\beta_2}^F\right]\right).$$

Then, for $\gamma \in \rho^{-1}(2(\lambda_1 + \lambda_2))_{\mathbb{R}}$, in the present situation (b) the vector $AW_\gamma$ is a *linear combination of brackets (evaluated at E) of fields in the distribution.*

7.3.3. *Situation (c).* We have

(c) $\alpha_{12}, \beta_{12} \in \rho^{-1}(\mu)_{\mathbb{R}}, \quad \alpha_1, \beta_1 \in \rho^{-1}(\eta)_{\mathbb{C}}^*, \quad \alpha_2, \beta_2 \in \rho^{-1}(\delta)_{\mathbb{C}}^*$.

We start by considering $\alpha_{12} \in \rho^{-1}(\mu)_{\mathbb{R}}$, with $\alpha_1 \in \rho^{-1}(\eta)_{\mathbb{C}}^*$ and $\alpha_2 \in \rho^{-1}(\delta)_{\mathbb{C}}^*$. Here we have to apply naturally formulae (6.8) but since $|\alpha_1 - \alpha_2|$ is not a root we see that this yields $H_1 = T_2 = 0$ (for $(\alpha_1, \alpha_2)$) and hence we may use (6.7) (with $(\eta, \delta, \alpha_1, \alpha_2)$ instead of $(\lambda, \mu, \delta, \varphi)$) which, replacing the roots, takes the form

$$\Theta_{(\eta,\delta,\alpha_1,\alpha_2)} W_{\alpha_{12}} = \begin{cases} \left[U_{\alpha_1}^F, U_{\alpha_2}^F\right](E) - \left[V_{\alpha_1}^F, V_{\alpha_2}^F\right](E) \\ \left[U_{\alpha_1}^F, V_{\alpha_2}^F\right](E) + \left[V_{\alpha_1}^F, U_{\alpha_2}^F\right](E). \end{cases}$$

This identity extends locally as we did to get (7.10) and recalling (7.8) we may write

$$aW^F_{\alpha_{12}} = \begin{cases} \left[U^F_{\alpha_1}, U^F_{\alpha_2}\right] - \left[V^F_{\alpha_1}, V^F_{\alpha_2}\right] \\ \left[U^F_{\alpha_1}, V^F_{\alpha_2}\right] + \left[V^F_{\alpha_1}, U^F_{\alpha_2}\right]. \end{cases} \tag{7.17}$$

where both sides are evaluated at the same point around $E$.

Proceeding similarly for $\beta_{12} \in \rho^{-1}(\mu)_{\mathbb{R}}$, $\beta_1 \in \rho^{-1}(\eta)^*_{\mathbb{C}}$, and $\beta_2 \in \rho^{-1}(\delta)^*_{\mathbb{C}}$ we have

$$bW^F_{\beta_{12}} = \begin{cases} \left[U^F_{\beta_1}, U^F_{\beta_2}\right] - \left[V^F_{\beta_1}, V^F_{\beta_2}\right] \\ \left[U^F_{\beta_1}, V^F_{\beta_2}\right] + \left[V^F_{\beta_1}, U^F_{\beta_2}\right]. \end{cases} \tag{7.18}$$

Since we are now in situation (c) we have we have to consider $\gamma = (\alpha_{12} + \beta_{12}) \in \rho^{-1}(2(\mu))_{\mathbb{R}}$, with $\alpha_{12} \in \rho^{-1}(\mu)_{\mathbb{R}}$ and $\beta_{12} \in \rho^{-1}(\mu)_{\mathbb{R}}$. Then we have again $\Lambda_{(\mu,\mu)} = 0$ and (6.6) lets us use (6.9) (with $(\mu, \mu, \alpha_{12}, \beta_{12})$ instead of $(\lambda, \mu, \delta, \varphi)$) and write

$$\Theta_{(\mu,\mu,\alpha_{12},\beta_{12})} W_\gamma = \pm \left[W^F_{\alpha_{12}}, W^F_{\beta_{12}}\right](E).$$

Proceeding as above we may replace each factor of the bracket $\left[aW^F_{\alpha_{12}}, bW^F_{\beta_{12}}\right]$ by any one of the two lines in (7.17) and (7.18) respectively, and expanding patiently (using (7.16)) we obtain the vector $AW_\gamma$ as a *linear combination of brackets (evaluated at $E$) of fields in the distribution.*

This completes the proof of Theorem 1.1.

## 8. Appendix

We include here as an example a proof of the lemmata in Section 4 for the spaces in Table 2. For simplicity we consider only the root system $(BC)_2$. Note that for *EIII* this is no restriction. We hope that this will serve as an example.

Table 2.

| **non-reduced** | **space** | $\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$ |
|---|---|---|
| *AIII* | $SU(p+2)/S(U(p) \times U(2))\ (p > 2)$ | $(BC)_2$ |
| *CII* | $Sp(p+2)/Sp(p) \times Sp(2)(p > 2)$ | $(BC)_2$ |
| *DIII* | $SO(2p)/U(p)$, $p = 5$, | $(BC)_2$ |
| *EIII* | $E_6/(Spin(10).T)$ | $(BC)_2$ |

In all cases we use the information from [2, p. 532–534]:

$$(BC)_2 \quad \Omega = \{\lambda_1, \lambda_2, \lambda_1 + 2\lambda_2\}, \quad \Gamma = \{\lambda_1 + \lambda_2, 2\lambda_2, 2\lambda_1 + 2\lambda_2\}.$$

The only root in $\Gamma$ which is not double is $\lambda = (\lambda_1 + \lambda_2)$; then taking $\eta = \lambda_1$ and $\delta = \lambda_2$ we have that $\lambda = \eta + \delta$ and $|\eta - \delta|$ is not a root of $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$, so the first part of Lemma 4.2 is clear.

8.1. **The space *AIII*.** To simplify notation we set $r = p + 3$; then $SU(p+2) = SU(r-1)$ and its Lie algebra is $A_r$, so the simple roots are $\{\alpha_1, \dots, \alpha_r\}$. The set of restricted roots is $(BC)_2$, so we have simple roots $\pi = \{\lambda_1, \lambda_2\}$. Since $p > 2$, we have $r > 5$. The restriction rule is

$$\begin{aligned} &\alpha_h \longmapsto \lambda_h, \quad \alpha_{r-h+1} \longmapsto \lambda_h, && 1 \le h \le 2, \\ &\alpha_k \longmapsto 0, && 3 \le k \le r-2, \end{aligned}$$

and the multiplicities are

$$m(\lambda_1) = 2, \quad m(\lambda_2) = 2(r-3), \quad m(2\lambda_2) = 1.$$

The roots going to $\lambda_1$ are $\{\alpha_1, \alpha_r\}$ and those with image $\lambda_2$ are

$$\begin{aligned} &\alpha_2 + \cdots + \alpha_k, && 3 \le k \le r-2; \\ &\alpha_k + \cdots + \alpha_{r-1}, && 3 \le k \le r-1. \end{aligned}$$

The root in $\Gamma$ which is not double is $\lambda = (\lambda_1 + \lambda_2)$ and this is image of the roots

$$\begin{aligned} &\alpha_1 + \alpha_2 + \cdots + \alpha_k, && 3 \le k \le r-2; \\ &\alpha_k + \cdots + \alpha_{r-1} + \alpha_r, && 3 \le k \le r-1, \end{aligned}$$

so it is clear that **for any** root $\gamma \in \rho^{-1}((\lambda_1 + \lambda_2)) \subset \Phi^+(\mathfrak{g}, \mathfrak{h})$ we can find $\alpha \in \rho^{-1}(\lambda_1)$ and $\beta \in \rho^{-1}(\lambda_2)$ such that $\gamma = \alpha + \beta$; in fact,

$$\begin{aligned} \alpha_1 + \alpha_2 + \cdots + \alpha_k &= (\alpha_1) + (\alpha_2 + \cdots + \alpha_k), && 3 \le k \le r-2; \\ \alpha_k + \cdots + \alpha_{r-1} + \alpha_r &= (\alpha_k + \cdots + \alpha_{r-1}) + (\alpha_r), && 3 \le k \le r-1. \end{aligned} \tag{8.1}$$

Then we have Lemma 4.2 for *AIII*.

The only root in $\Omega$ whose double is in $\Gamma$ is $\lambda_2$. Then since $r > 5$ we have that $\left|\rho^{-1}(\lambda_2)\right| = 2(r-3) > 4$ and **there is only one root in** $\gamma \in \rho^{-1}(2\lambda_2)$; in fact it is $\gamma = (\alpha_2 + \cdots + \alpha_{r-1})$, so we can chose two **different** roots $\alpha$, $\beta \in \rho^{-1}(\lambda_2)$ such that $\gamma = \alpha + \beta$. In fact we may take

$$\alpha = \alpha_2 + \cdots + \alpha_k, \quad \beta = \alpha_{k+1} + \cdots + \alpha_{r-1}, \qquad 2 < k < r-1.$$

Then we have Lemma 4.3 for *AIII*.

Now consider the double root $\lambda = 2(\lambda_1 + \lambda_2)$, so $\gamma = 2\mu$, with $\mu = (\lambda_1 + \lambda_2)$. We have to find $\eta \neq \delta$ in $\Omega$ such that $\mu = (\lambda_1 + \lambda_2) = \eta + \delta$ and $|\eta - \delta|$ is not a root of $(BC)_q$. Obviously we have to take $\eta = \lambda_1$ and $\delta = \lambda_2$.

There is only one root in $\rho^{-1}(\lambda)$, namely $\gamma = (\alpha_1 + \cdots + \alpha_r)$. Now we have to find $\varphi_1 \neq \beta_1 \in \rho^{-1}(\lambda_1)$ and $\varphi_2 \neq \beta_2 = \rho^{-1}(\lambda_2)$ such that $\varphi_1 + \varphi_2$ and $\beta_1 + \beta_2$

belongs to $\rho^{-1}(\mu) = \rho^{-1}(\lambda_1 + \lambda_2)$ and $\gamma = \alpha_1 + \alpha_2 + \beta_1 + \beta_2$. We take

$$\begin{aligned}
&\varphi_1 = \alpha_1, \\
&\beta_1 = \alpha_r, \\
&\varphi_1 \neq \beta_1 \in \rho^{-1}(\lambda_1), \\
&\varphi_2 = (\alpha_2 + \cdots + \alpha_k), && 3 \leq k \leq r-2, \\
&\beta_2 = (\alpha_{k+1} + \cdots + \alpha_{r-1}), && 3 \leq k \leq r-1, \\
&\varphi_2 \neq \beta_2 \in \rho^{-1}(\lambda_2).
\end{aligned}$$

We have, by (8.1),

$$\begin{aligned}
\varphi_1 + \varphi_2 &= (\alpha_1) + (\alpha_2 + \cdots + \alpha_k) \in \rho^{-1}(\lambda_1 + \lambda_2), \\
\beta_1 + \beta_2 &= (\alpha_r) + (\alpha_{k+1} + \cdots + \alpha_{r-1}) \in \rho^{-1}(\lambda_1 + \lambda_2),
\end{aligned}$$

and furthermore

$$\begin{aligned}
\gamma &= (\alpha_1 + \cdots + \alpha_r) \\
&= (\alpha_1) + (\alpha_2 + \cdots + \alpha_k) + (\alpha_{k+1} + \cdots + \alpha_{r-1}) + (\alpha_r) \\
&= \varphi_1 + \varphi_2 + \beta_1 + \beta_2.
\end{aligned}$$

Then we have Lemma 4.4 for *AIII*.

8.2. **The space *CII* ($p > 2$).** For $p > 2$ the restricted roots form $(BC)_2$. Let us set $r = p + 2$. The algebra of $Sp(r)$ is $C_r$. Since $p > 2$ we have here $r > 4$.

Again the system of simple roots $\pi = \{\alpha_1, \ldots, \alpha_r\}$ and the roots of $C_r$ are

$$\begin{aligned}
&(1)\ \ e_i - e_j = \sum_{i \leq k < j} \alpha_k, && 1 \leq i < j \leq r, \\
&(2)\ \ e_i + e_j = \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k < r} \alpha_k + \alpha_r, && 1 \leq i < j \leq r, \\
&(3)\ \ 2e_i = 2 \sum_{i \leq k < r} \alpha_k + \alpha_r, && 1 \leq i \leq r.
\end{aligned} \tag{8.2}$$

We observe that for those roots of type (2) in (8.2) we have the particular case $j = r$ which are the roots

$$e_i + e_r = \sum_{i \leq k < r} \alpha_k + \alpha_r, \quad 1 \leq i < r.$$

The restriction rule is

$$\begin{aligned}
\alpha_{2j} &\longmapsto \lambda_j, && \text{if } 1 \leq j \leq 2, \\
\alpha_k &\longmapsto 0, && \text{if either } k \text{ is odd or } k > 4.
\end{aligned}$$

The multiplicities are

$$m(\lambda_1) = 4, \quad m(\lambda_2) = 4(r-4), \quad m(2\lambda_2) = 3.$$

The roots in $\rho^{-1}(\lambda_1)$ are the four roots

$$\alpha_2 = (e_2 - e_3), \qquad \alpha_1 + \alpha_2 = (e_1 - e_3),$$
$$\alpha_2 + \alpha_3 = (e_2 - e_4), \qquad \alpha_1 + \alpha_2 + \alpha_3 = (e_1 - e_4).$$

We have in $\rho^{-1}(\lambda_2)$ the roots

$$\begin{aligned}
&e_4 - e_{h+1} = \alpha_4 + \cdots + \alpha_h,\\
&e_3 - e_{h+1} = \alpha_3 + \alpha_4 + \cdots + \alpha_h,\\
&e_4 + e_{h+1} = \alpha_4 + \cdots + \alpha_h + 2(\alpha_{h+1} + \cdots + \alpha_{r-1}) + \alpha_r,\\
&e_3 + e_{h+1} = \alpha_3 + \alpha_4 + \cdots + \alpha_h + 2(\alpha_{h+1} + \cdots + \alpha_{r-1}) + \alpha_r,\\
&4 \leq h \leq r-1,
\end{aligned}$$

(if $h = r-1$ the factor of 2 is not present). The multiplicity of $\lambda_q$ is $m(\lambda_q) = 4(r-4)$.

The only root in $\Gamma$ which is not double is $\lambda = (\lambda_1 + \lambda_2)$ and it is image of the roots that contain $\alpha_2$ and $\alpha_4$ as terms with coefficient 1. They are

$$\begin{aligned}
&(1)\ \ e_i - e_j = \sum_{i\leq k<j} \alpha_k, && i = 1,2;\ 3 \leq j \leq r,\\
&(2)\ \ e_i + e_j = \sum_{i\leq k<j} \alpha_k + 2\sum_{j\leq k<r} \alpha_k + \alpha_r, && i = 1,2;\ 3 \leq j \leq r-1,\\
&(2'')\ \ e_i + e_r = \sum_{i\leq k<r} \alpha_k + \alpha_r, && i = 1,2,
\end{aligned}$$

which we simplify to

$$\begin{aligned}
&(e_1 - e_h),\ (e_2 - e_h), && 3 \leq h \leq r,\\
&(e_1 + e_h),\ (e_2 + e_h), && 3 \leq h \leq r-1,\\
&(e_1 + e_r), (e_2 + e_r), && m(\lambda_1 + \lambda_2) = 4(r-3).
\end{aligned}$$

We may write them as

$$\begin{aligned}
&(e_1 - e_h) = (\alpha_1 + \alpha_2) + (e_3 - e_h) = (e_1 - e_3) + (e_3 - e_h), && 3 \leq h \leq r,\\
&(e_2 - e_h) = (\alpha_2 + \alpha_3) + (e_4 - e_h) = (e_2 - e_4) + (e_4 - e_h), && 3 \leq h \leq r,\\
&(e_1 + e_h) = (\alpha_1 + \alpha_2) + (e_3 + e_h) = (e_1 - e_3) + (e_3 + e_h), && 3 \leq h \leq r-1,\\
&(e_2 + e_h) = (\alpha_2 + \alpha_3) + (e_4 + e_h) = (e_2 - e_4) + (e_4 + e_h), && 3 \leq h \leq r-1,\\
&(e_1 + e_r) = (\alpha_1 + \alpha_2) + (e_3 + e_h) = (e_1 - e_3) + (e_3 + e_h),\\
&(e_2 + e_r) = (\alpha_2) + (e_3 + e_h) = (e_2 - e_3) + (e_3 + e_h).
\end{aligned}$$

Then **for any** root $\gamma \in \rho^{-1}((\lambda_1 + \lambda_2))$ there exist roots $\varphi \in \rho^{-1}(\lambda_1)$ and $\beta \in \rho^{-1}(\lambda_2)$ such that $\gamma = \alpha + \beta$.

Then we have Lemma 4.2 for *CII*.

**Now we prove Lemma 4.3.**

Assume that $\lambda \in \Gamma$, $\lambda = 2\mu$, and $\mu \in \Omega$, and let $\gamma \in \rho^{-1}(\lambda)$. Then there exist roots $\alpha$, $\beta \in \rho^{-1}(\mu)$ such that $\alpha \neq \beta$ and $\gamma = \alpha + \beta$.

The roots in $\rho^{-1}(2\lambda_2)$ are the **three roots**

$$\begin{aligned} \xi_1 &= e_3 + e_4 = \alpha_3 + 2\sum_{4\leq k<r} \alpha_k + \alpha_r, \\ \xi_2 &= 2e_4 = 2\sum_{4\leq k<r} \alpha_k + \alpha_r, \\ \xi_3 &= 2e_3 = 2\sum_{3\leq k<r} \alpha_k + \alpha_r, \end{aligned} \tag{8.3}$$

so we have $m(2\lambda_2) = 3$. Let us take now $\xi_1 \in \rho^{-1}(2\lambda_2)$ in (8.3). We have to show that there exist roots $\eta_1, \beta_1 \in \rho^{-1}(\lambda_2)$ such that $\eta_1 \neq \beta_1$ and $\xi_1 = \eta_1 + \beta_1$. We define, for

$$\xi_1 = e_3 + e_4 = \alpha_3 + 2\sum_{4\leq k<r} \alpha_k + \alpha_r,$$

the roots

$$\begin{aligned} \eta_1 &= e_3 - e_{r-1} = \alpha_3 + \sum_{4\leq s\leq r-1} \alpha_s, \\ \beta_1 &= e_4 + e_{r-1} = \sum_{4\leq s\leq r-1} \alpha_s + \alpha_r. \end{aligned}$$

Clearly $\eta_1 \neq \beta_1$ and

$$\eta_1 + \beta_1 = (e_{2j-1} - e_{r-1}) + (e_{2j} + e_{r-1}) = e_{2j-1} + e_{2j} = \xi_1.$$

Now, for

$$\xi_2 = 2e_4 = 2\sum_{4\leq k<r} \alpha_k + \alpha_r,$$

we may take

$$\begin{aligned} \eta_2 &= e_4 - e_{r-1} = \sum_{2j\leq s\leq r-1} \alpha_s, \\ \beta_2 &= e_4 + e_{r-1} = \sum_{2j\leq s\leq r-1} \alpha_s + \alpha_r; \end{aligned}$$

then again $\eta_2 \neq \beta_2$ and

$$\eta_2 + \beta_2 = (e_{2j} - e_{r-1}) + (e_{2j} + e_{r-1}) = 2e_{2j} = \xi_2.$$

Now, for

$$\xi_3 = 2e_3 = 2\sum_{3\leq k<r} \alpha_k + \alpha_r,$$

we take

$$\begin{aligned} \eta_3 &= e_3 - e_{r-1} = \sum_{(2j-1)\leq s\leq r-1} \alpha_s, \\ \beta_2 &= e_3 + e_{r-1} = \sum_{(2j-1)\leq s\leq r-1} \alpha_s + \alpha_r, \end{aligned}$$

and, once again, $\eta_3 \neq \beta_3$ and

$$\eta_3 + \beta_3 = (e_{2j-1} - e_{r-1}) + (e_{2j-1} + e_{r-1}) = 2e_{2j-1} = \xi_3.$$

Furthermore, it is clear by the definitions of $\eta_s$ and $\beta_s$ (for $1 \leq s \leq 3$) that we also have

$$\rho(\eta_s) = \lambda_2 = \rho(\beta_s).$$

Then we have Lemma 4.3 for *CII*.

**Now we prove Lemma 4.4.**

We take $\lambda = 2(\lambda_1 + \lambda_2)$ ($\lambda \in \Gamma$, $\lambda = 2\mu$, and $\mu \in \Gamma$ with $\mu = (\lambda_1 + \lambda_2)$). The roots in $\rho^{-1}(2(\lambda_1 + \lambda_2))$ are the three roots

$$\begin{aligned}
&\xi_1 = e_1 + e_2 = \alpha_1 + 2\sum_{2 \leq k < r} \alpha_k + \alpha_r,\\
&\xi_2 = 2e_2 = 2\sum_{2 \leq k < r} \alpha_k + \alpha_r,\\
&\xi_3 = 2e_1 = 2\sum_{1 \leq k < r} \alpha_k + \alpha_r,\\
&m(2(\lambda_1 + \lambda_2)) = 3.
\end{aligned}$$

We set now for $\xi_1 = e_1 + e_2$:

$$\begin{aligned}
&\omega_1 = e_2 - e_4, && \beta_1 = e_1 - e_4,\\
&\omega_2 = e_4 + e_r, && \beta_2 = e_4 - e_r,\\
&\omega_1 \neq \beta_1 \in \rho^{-1}(\lambda_1), && \omega_2 \neq \beta_2 \in \rho^{-1}(\lambda_2).
\end{aligned}$$

Then we have

$$\begin{aligned}
\omega_1 + \omega_2 &= e_2 + e_r \in \rho^{-1}(\lambda_1 + \lambda_2),\\
\beta_1 + \beta_2 &= e_1 - e_r \in \rho^{-1}(\lambda_1 + \lambda_2),
\end{aligned}$$

and furthermore

$$\omega_1 + \beta_1 + \omega_2 + \beta_2 = (e_2 + e_r) + (e_1 - e_r) = e_1 + e_2 = \xi_1.$$

Considering now $\xi_2 = 2e_2$ we set

$$\begin{aligned}
&\omega_1 = (\alpha_2) = e_2 - e_3, && \beta_1 = (\alpha_2 + \alpha_3) = e_2 - e_4,\\
&\omega_2 = e_3 + e_r, && \beta_2 = e_4 - e_r,\\
&\omega_1 \neq \beta_1 \in \rho^{-1}(\lambda_1), && \omega_2 \neq \beta_2 \in \rho^{-1}(\lambda_2).
\end{aligned}$$

Then we have

$$\begin{aligned}
\omega_1 + \omega_2 &= e_2 + e_r \in \rho^{-1}(\lambda_1 + \lambda_2),\\
\beta_1 + \beta_2 &= e_2 - e_r \in \rho^{-1}(\lambda_1 + \lambda_2),
\end{aligned}$$

and finally

$$\omega_1 + \omega_2 + \beta_1 + \beta_2 = (e_2 + e_r) + (e_2 - e_r) = 2e_2 = \xi_2.$$

Now for $\xi_3 = 2e_1$ we define

$$\begin{aligned} \omega_1 &= (\alpha_1+\alpha_2) = e_1 - e_3, & \omega_2 &= e_3 + e_r, \\ \beta_1 &= (\alpha_1+\alpha_2+\alpha_3) = e_1 - e_4, & \beta_2 &= e_4 - e_r, \\ \omega_1 &\neq \beta_1 \in \rho^{-1}(\lambda_1), & \omega_2 &\neq \beta_2 \in \rho^{-1}(\lambda_2). \end{aligned}$$

Then we have

$$\begin{aligned} \omega_1 + \omega_2 &= e_1 + e_r \in \rho^{-1}(\lambda_1+\lambda_2), \\ \beta_1 + \beta_2 &= e_1 - e_r \in \rho^{-1}(\lambda_1+\lambda_2), \end{aligned}$$

and finally

$$\omega_1 + \omega_2 + \beta_1 + \beta_2 = (e_1+e_r) + (e_1 - e_r) = 2e_1 = \xi_3.$$

Then we have Lemma 4.4 for *CII*.

8.3. **The space *DIII* ($p = 5$).** The system of roots for $SO(10)$ is $D_5$. In the present case the restriction rule is

$$\alpha_2 \longmapsto \lambda_1, \quad (\alpha_4,\alpha_5) \longmapsto \lambda_2, \quad (\alpha_1,\alpha_3) \longmapsto 0,$$

and the multiplicities are

$$m(\lambda_j) = 4, \quad j = 1,2; \qquad m(2\lambda_2) = 1.$$

We need to indicate the roots of $D_5$ in the notation from Bourbaki. If $\{e_j\}$ is the canonical basis of $\mathbb{R}^5$ the roots are

$$\pm e_j \pm e_k, \quad 1 \leq j < k \leq p.$$

The number of positive roots is 20 and the basis is

$$\begin{aligned} \alpha_1 &= e_1 - e_2, & \alpha_2 &= e_2 - e_3, & \alpha_3 &= e_3 - e_4, \\ \alpha_4 &= e_4 - e_5, & \alpha_5 &= e_4 + e_5. \end{aligned}$$

The positive roots for $D_5$ are

$$\begin{aligned} &(1)\ e_i - e_j = \sum_{i\leq u<j} \alpha_u, && 1 \leq i < j \leq 5; \\ &(2)\ e_i - e_5 = \sum_{i\leq u<5} \alpha_u, && 1 \leq i < 5; \\ &(3)\ e_i + e_5 = \sum_{i\leq u<4} \alpha_u + \alpha_5, && 1 \leq i < 4; \\ &(4)\ e_i + e_j = \sum_{i\leq u<j} \alpha_u + 2\sum_{j\leq u<4} \alpha_u + \alpha_4 + \alpha_5, && 1 \leq i < j < 4. \end{aligned}$$

The roots that go to $\lambda_1$ and $\lambda_2$ are respectively the four roots

$$\rho^{-1}(\lambda_1) = \left\{ \begin{array}{ll} \alpha_2 = (e_2 - e_3) & \alpha_1 + \alpha_2 = (e_1 - e_3) \\ \alpha_2 + \alpha_3 = (e_2 - e_4) & \alpha_1 + \alpha_2 + \alpha_3 = (e_1 - e_4) \end{array} \right\}$$

$$\rho^{-1}(\lambda_2) = \left\{ \begin{array}{ll} \alpha_4 = (e_4 - e_5) & \alpha_5 = (e_4 + e_5) \\ \alpha_3 + \alpha_4 = (e_3 - e_5) & \alpha_3 + \alpha_5 = (e_3 + e_5) \end{array} \right\}.$$

**Now we consider the roots in** $\rho^{-1}(\lambda_1+\lambda_2)$. These are the roots containing $\alpha_2$ and $\alpha_4$ *with coefficient* 1 or $\alpha_2$ and $\alpha_5$ also *with coefficient* 1, so they are

$$\begin{aligned} e_2-e_5 &= \alpha_2+\alpha_3+\alpha_4\\ e_1-e_5 &= \alpha_1+\alpha_2+\alpha_3+\alpha_4\\ e_2+e_5 &= \alpha_2+\alpha_3+\alpha_5\\ e_1+e_5 &= \alpha_1+\alpha_2+\alpha_3+\alpha_5. \end{aligned}$$

They may be writen as

$$\begin{aligned} e_2-e_5 &= \alpha_2+\alpha_3+\alpha_4 = (e_2-e_3)+(e_3-e_5)\\ e_1-e_5 &= \alpha_1+\alpha_2+\alpha_3+\alpha_4 = (e_1-e_4)+(e_4-e_5)\\ e_2+e_5 &= \alpha_2+\alpha_3+\alpha_5 = (e_2-e_4)+(e_4+e_5)\\ e_1+e_5 &= \alpha_1+\alpha_2+\alpha_3+\alpha_5 = (e_1-e_4)+(e_4+e_5)\,. \end{aligned}$$

Then *for any* root $\gamma\in\rho^{-1}((\lambda_1+\lambda_2))\subset\Phi^+(\mathfrak{g},\mathfrak{h})$ there exist roots $\varphi\in\rho^{-1}(\lambda_1)$ and $\beta\in\rho^{-1}(\lambda_2)$ such that $\gamma=\varphi+\beta$.

Then we have Lemma 4.2 for *DIII*.

**Now we consider the root in** $\rho^{-1}(2\lambda_2)$**.** There is only one:

$$\gamma=e_3+e_4=\alpha_3+\alpha_4+\alpha_5\in\rho^{-1}(2\lambda_2).$$

To prove Lemma 4.3 for *DIII* we take $\gamma$ and have to find $\alpha$, $\beta\in\rho^{-1}(\lambda_2)$ such that $\alpha\neq\beta$ and $\gamma=\alpha+\beta$. Then we may take

$$\alpha=\alpha_5,\qquad \beta=\alpha_3+\alpha_4.$$

Then we have Lemma 4.3 for *DIII*.

**Now we consider the roots in** $\rho^{-1}(2(\lambda_1+\lambda_2))$**.** There is only one:

$$\rho^{-1}(2(\lambda_1+\lambda_2))=\{\alpha_1+2(\alpha_2+\alpha_3)+\alpha_4+\alpha_5\}\,.$$

Here $2(\lambda_1+\lambda_2)\in\Gamma$ and $\mu=(\lambda_1+\lambda_2)\in\Gamma$. Since $\mu$ is not a double root, by Lemma 4.1 we have two roots $\lambda_1\neq\lambda_2$ in $\Omega$, such that $\mu=(\lambda_1+\lambda_2)$ and $|\lambda_1-\lambda_2|$ is not a root of $(BC)_q$. Then *for any* root $\gamma\in\rho^{-1}(2(\lambda_1+\lambda_2))$ (only one here) there exist roots $\varphi_1\neq\beta_1\in\rho^{-1}(\lambda_1)$ and $\varphi_2\neq\beta_2\in\rho^{-1}(\lambda_2)$ such that $\varphi_1+\varphi_2$ and $\beta_1+\beta_2$ belong to $\rho^{-1}((\lambda_1+\lambda_2))$, and furthermore $\gamma=\varphi_1+\varphi_2+\beta_1+\beta_2$. We take

$$\begin{aligned} &\varphi_1=\alpha_1+\alpha_2+\alpha_3=(e_1-e_4) && \beta_1=\alpha_2+\alpha_3=(e_2-e_4)\\ &\varphi_1\neq\beta_1\in\rho^{-1}(\lambda_1) &&\\ &\varphi_2=\alpha_4=(e_4-e_5) && \beta_2=\alpha_5=(e_4+e_5)\\ &\varphi_2\neq\beta_2\in\rho^{-1}(\lambda_2). && \end{aligned}$$

We clearly have

$$\begin{aligned}\varphi_1 + \varphi_2 &= (e_1 - e_4) + (e_4 - e_5) \\ &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = (e_1 - e_5) \in \rho^{-1}(\lambda_1 + \lambda_2),\end{aligned}$$

$$\begin{aligned}\beta_1 + \beta_2 &= (e_2 - e_4) + (e_4 + e_5) \\ &= \alpha_2 + \alpha_3 + \alpha_5 = (e_2 + e_5) \in \rho^{-1}(\lambda_1 + \lambda_2),\end{aligned}$$

and furthermore

$$\begin{aligned}\varphi_1 + \varphi_2 + \beta_1 + \beta_2 &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + (\alpha_2 + \alpha_3 + \alpha_5) \\ &= \alpha_1 + 2\,(\alpha_2 + \alpha_3) + \alpha_4 + \alpha_5.\end{aligned}$$

Then we have Lemma 4.4 for *DIII*.

8.4. **The space *EIII*.** We need the restriction rule from [2, p. 534]. We keep the previous notation for $(BC)_2$ but this means a change in names of the simple roots from the indicated in the diagram in [2, p. 534].

$$\boxed{\begin{array}{ccccc} & & \alpha_2 & & \\ \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_1 \end{array}} = \boxed{\begin{array}{ccccc} & & \alpha_2 & & \\ \alpha_6 & \bullet & \bullet & \bullet & \alpha_1 \end{array}} \longmapsto (\lambda_1, \lambda_2)$$

$$\alpha_2 \longmapsto \lambda_1, \quad \alpha_6, \alpha_1 \longmapsto \lambda_2, \quad \alpha_3, \alpha_4, \alpha_5 \longmapsto 0,$$

and the multiplicities are

$$m(\lambda_1) = 6, \quad m(\lambda_2) = 8, \quad m(2\lambda_2) = 1.$$

The roots in $\rho^{-1}(\lambda_1)$ and $\rho^{-1}(\lambda_2)$ are:

$$\boxed{\begin{array}{ccccc} & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{array}} \longmapsto \lambda_1, \quad \boxed{\begin{array}{ccccc} & & 1 & & \\ 0 & 0 & 1 & 0 & 0 \end{array}} \longmapsto \lambda_1$$

$$\boxed{\begin{array}{ccccc} & & 1 & & \\ 0 & 1 & 1 & 0 & 0 \end{array}} \longmapsto \lambda_1, \quad \boxed{\begin{array}{ccccc} & & 1 & & \\ 0 & 0 & 1 & 1 & 0 \end{array}} \longmapsto \lambda_1$$

$$\boxed{\begin{array}{ccccc} & & 1 & & \\ 0 & 1 & 1 & 1 & 0 \end{array}} \longmapsto \lambda_1, \quad \boxed{\begin{array}{ccccc} & & 1 & & \\ 0 & 1 & 2 & 1 & 0 \end{array}} \longmapsto \lambda_1$$

$$\boxed{\begin{array}{ccccc} & & 0 & & \\ 1 & 0 & 0 & 0 & 0 \end{array}} \longmapsto \lambda_2, \quad \boxed{\begin{array}{ccccc} & & 0 & & \\ 0 & 0 & 0 & 0 & 1 \end{array}} \longmapsto \lambda_2$$

$$\boxed{\begin{array}{ccccc} & & 0 & & \\ 1 & 1 & 0 & 0 & 0 \end{array}} \longmapsto \lambda_2, \quad \boxed{\begin{array}{ccccc} & & 0 & & \\ 0 & 0 & 0 & 1 & 1 \end{array}} \longmapsto \lambda_2$$

$$\boxed{\begin{array}{ccccc} & & 0 & & \\ 1 & 1 & 1 & 0 & 0 \end{array}} \longmapsto \lambda_2, \quad \boxed{\begin{array}{ccccc} & & 0 & & \\ 0 & 0 & 1 & 1 & 1 \end{array}} \longmapsto \lambda_2$$

$$\boxed{\begin{array}{ccccc} & & 0 & & \\ 1 & 1 & 1 & 1 & 0 \end{array}} \longmapsto \lambda_2, \quad \boxed{\begin{array}{ccccc} & & 0 & & \\ 0 & 1 & 1 & 1 & 1 \end{array}} \longmapsto \lambda_2$$

The roots in $\rho^{-1}(\lambda_1+\lambda_2)$ are:

$$\boxed{\begin{smallmatrix} &&1&& \\ 1&1&1&0&0 \end{smallmatrix}} \longmapsto \lambda_1+\lambda_2, \quad \boxed{\begin{smallmatrix} &&1&& \\ 0&0&1&1&1 \end{smallmatrix}} \longmapsto \lambda_1+\lambda_2$$

$$\boxed{\begin{smallmatrix} &&1&& \\ 1&1&1&1&0 \end{smallmatrix}} \longmapsto \lambda_1+\lambda_2, \quad \boxed{\begin{smallmatrix} &&1&& \\ 0&1&1&1&1 \end{smallmatrix}} \longmapsto \lambda_1+\lambda_2$$

$$\boxed{\begin{smallmatrix} &&1&& \\ 1&1&2&1&0 \end{smallmatrix}} \longmapsto \lambda_1+\lambda_2, \quad \boxed{\begin{smallmatrix} &&1&& \\ 0&1&2&1&1 \end{smallmatrix}} \longmapsto \lambda_1+\lambda_2$$

$$\boxed{\begin{smallmatrix} &&1&& \\ 1&2&2&1&0 \end{smallmatrix}} \longmapsto \lambda_1+\lambda_2, \quad \boxed{\begin{smallmatrix} &&1&& \\ 0&1&2&2&1 \end{smallmatrix}} \longmapsto \lambda_1+\lambda_2$$

There is only one root in $\rho^{-1}(2\lambda_2)$ and also one in $\rho^{-1}(2\lambda_2+2\lambda_1)$. They are:

$$\boxed{\begin{smallmatrix} &&0&& \\ 1&1&1&1&1 \end{smallmatrix}} \longmapsto 2\lambda_2$$

$$\boxed{\begin{smallmatrix} &&2&& \\ 1&2&3&2&1 \end{smallmatrix}} \longmapsto 2\lambda_2+2\lambda_1.$$

In order to prove Lemma 4.2 let us take now $(\lambda_1+\lambda_2)$. We have to show that for any root $\gamma\in\rho^{-1}(\lambda_1+\lambda_2)\subset\Phi^+(\mathfrak{g},\mathfrak{h})$ there exist roots $\alpha\in\rho^{-1}(\lambda_1)$ and $\beta\in\rho^{-1}(\lambda_2)$ such that $\gamma=\alpha+\beta$. We can write each root going to $\lambda_1+\lambda_2$ as a sum of one going to $\lambda_2$ plus one going to $\lambda_1$.

(1) $(\lambda_1+\lambda_2)$ $\boxed{\begin{smallmatrix} &&1&& \\ 1&1&1&0&0 \end{smallmatrix}} = \boxed{\begin{smallmatrix} &&0&& \\ 1&1&1&0&0 \end{smallmatrix}} + \boxed{\begin{smallmatrix} &&1&& \\ 0&0&0&0&0 \end{smallmatrix}}$

(2) $(\lambda_1+\lambda_2)$ $\boxed{\begin{smallmatrix} &&1&& \\ 0&0&1&1&1 \end{smallmatrix}} = \boxed{\begin{smallmatrix} &&0&& \\ 0&0&1&1&1 \end{smallmatrix}} + \boxed{\begin{smallmatrix} &&1&& \\ 0&0&0&0&0 \end{smallmatrix}}$

(3) $(\lambda_1+\lambda_2)$ $\boxed{\begin{smallmatrix} &&1&& \\ 1&1&1&1&0 \end{smallmatrix}} = \boxed{\begin{smallmatrix} &&0&& \\ 1&1&1&1&0 \end{smallmatrix}} + \boxed{\begin{smallmatrix} &&1&& \\ 0&0&0&0&0 \end{smallmatrix}}$

(4) $(\lambda_1+\lambda_2)$ $\boxed{\begin{smallmatrix} &&1&& \\ 0&1&1&1&1 \end{smallmatrix}} = \boxed{\begin{smallmatrix} &&0&& \\ 0&1&1&1&1 \end{smallmatrix}} + \boxed{\begin{smallmatrix} &&1&& \\ 0&0&0&0&0 \end{smallmatrix}}$

(5) $(\lambda_1+\lambda_2)$ $\boxed{\begin{smallmatrix} &&1&& \\ 1&1&2&1&0 \end{smallmatrix}} = \boxed{\begin{smallmatrix} &&0&& \\ 1&0&0&0&0 \end{smallmatrix}} + \boxed{\begin{smallmatrix} &&1&& \\ 0&1&2&1&0 \end{smallmatrix}}$

(6) $(\lambda_1+\lambda_2)$ $\boxed{\begin{smallmatrix} &&1&& \\ 0&1&2&1&1 \end{smallmatrix}} = \boxed{\begin{smallmatrix} &&0&& \\ 0&0&0&0&1 \end{smallmatrix}} + \boxed{\begin{smallmatrix} &&1&& \\ 0&1&2&1&0 \end{smallmatrix}}$

(7) $(\lambda_1+\lambda_2)$ $\boxed{\begin{smallmatrix} &&1&& \\ 1&2&2&1&0 \end{smallmatrix}} = \boxed{\begin{smallmatrix} &&0&& \\ 1&1&0&0&0 \end{smallmatrix}} + \boxed{\begin{smallmatrix} &&1&& \\ 0&1&2&1&0 \end{smallmatrix}}$

(8) $(\lambda_1+\lambda_2)$ $\boxed{\begin{smallmatrix} &&1&& \\ 0&1&2&2&1 \end{smallmatrix}} = \boxed{\begin{smallmatrix} &&0&& \\ 0&0&0&1&1 \end{smallmatrix}} + \boxed{\begin{smallmatrix} &&1&& \\ 0&1&2&1&0 \end{smallmatrix}}$

Therefore we have shown that, for all roots $\gamma\in\Phi^+(\mathfrak{g},\mathfrak{h})$ such that $\rho(\gamma)=(\lambda_1+\lambda_2)\in\Gamma$, there exist two roots $\alpha$ and $\beta\in\Phi^+(\mathfrak{g},\mathfrak{h})$ such that $\rho(\alpha)$ and $\rho(\beta)\in\Omega$ and $\gamma=\alpha+\beta$.

Then we have Lemma 4.2 for *EIII*.

Let us take now the proof of Lemma 4.3.

We consider the double root $2\lambda_2$ ($\lambda_2 \in \Omega$) and let $\gamma \in \rho^{-1}(2\lambda_2)$. We must show that there exist roots $\varphi$, $\beta \in \rho^{-1}(\mu)$ such that $\alpha \neq \beta$ and $\gamma = \alpha + \beta$. Since there is only one root in $\rho^{-1}(2\lambda_2)$, this can be done by

$$\boxed{\begin{array}{ccccc} & & 0 & & \\ 1 & 1 & 1 & 1 & 1 \end{array}} = \boxed{\begin{array}{ccccc} & & 0 & & \\ 1 & 0 & 0 & 0 & 0 \end{array}} + \boxed{\begin{array}{ccccc} & & 0 & & \\ 0 & 1 & 1 & 1 & 1 \end{array}}.$$

Then we have Lemma 4.3 for *EIII*.

Finally we have to prove that Lemma 4.4 holds for *EIII* and then we take $2\lambda_2 + 2\lambda_1$. There is only one root in $\rho^{-1}(2(\lambda_1 + \lambda_2))$:

$$\boxed{\begin{array}{ccccc} & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{array}} \longmapsto 2\lambda_2 + 2\lambda_1.$$

Here $\lambda = 2\mu = 2(\lambda_1 + \lambda_2)$ and $\mu = (\lambda_1 + \lambda_2)$, $\lambda_1 \neq \lambda_2$ in $\Omega$, $|\lambda_1 - \lambda_2|$ is not a root of $(BC)_2$. We have to show that *for any* root $\gamma \in \rho^{-1}\left(2\left(\lambda_1 + \lambda_2\right)\right)$ there exist roots $\varphi_1 \neq \beta_1 \in \rho^{-1}\left(\lambda_1\right)$ and $\varphi_2 \neq \beta_2 \in \rho^{-1}\left(\lambda_2\right)$ such that $\varphi_1 + \varphi_2$ and $\beta_1 + \beta_2$ belong to $\rho^{-1}(\mu) = \rho^{-1}\left(\lambda_1 + \lambda_2\right)$, and furthermore $\gamma = \varphi_1 + \varphi_2 + \beta_1 + \beta_2$. There is one root in $\rho^{-1}(2\lambda_2 + 2\lambda_1)$ indicated above. We take

$$\varphi_1 = \boxed{\begin{array}{ccccc} & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{array}} \longmapsto \lambda_1, \quad \beta_1 = \boxed{\begin{array}{ccccc} & & 1 & & \\ 0 & 1 & 2 & 1 & 0 \end{array}} \longmapsto \lambda_1$$

$\varphi_1 \neq \beta_1 \in \rho^{-1}(\lambda_1)$

$$\varphi_2 = \boxed{\begin{array}{ccccc} & & 0 & & \\ 1 & 1 & 1 & 0 & 0 \end{array}} \longmapsto \lambda_2, \quad \beta_2 = \boxed{\begin{array}{ccccc} & & 0 & & \\ 0 & 0 & 0 & 1 & 1 \end{array}} \longmapsto \lambda_2$$

$\varphi_2 \neq \beta_2 \in \rho^{-1}(\lambda_2)$

$$\boxed{\begin{array}{ccccc} & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{array}} = \boxed{\begin{array}{ccccc} & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{array}} + \boxed{\begin{array}{ccccc} & & 1 & & \\ 0 & 1 & 2 & 1 & 0 \end{array}}$$
$$+ \boxed{\begin{array}{ccccc} & & 0 & & \\ 1 & 1 & 1 & 0 & 0 \end{array}} + \boxed{\begin{array}{ccccc} & & 0 & & \\ 0 & 0 & 0 & 1 & 1 \end{array}}.$$

Then we have Lemma 4.4 for *EIII*.

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