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LINEAR MAPS PRESERVING DRAZIN INVERSES OF MATRICES OVER LOCAL RINGS

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Abstract. Let R be a local ring and suppose that there exists $a \in F^*$ such that $a^6 \neq 1$; also let $T : M_n(R) \to M_m(R)$ be a linear map preserving Drazin inverses. Then we prove that $T = 0$ or $n = m$ and T preserves idempotents. We thereby determine the form of linear maps from $M_n(R)$ to $M_m(R)$ preserving Drazin inverses of matrices.

1. Introduction

Let R be a commutative ring with an identity. $M_n(R)$ denotes the $n \times n$ matrix algebra over R and $GL_n(R)$ stands for the general linear group of $M_n(R)$ for a positive integer n. A matrix $A \in M_n(R)$ has Drazin inverse if there exists $B \in M_n(R)$ such that

$$B = BAB, \quad AB = BA, \quad A^k = A^{k+1}B \quad \text{for some } k \in \mathbb{N}.$$

The preceding B is unique if it exists; we denote it by A^D. The Drazin inverse plays an important role in matrix and operator theory (see [7, 12, 13, 15, 14]). We say that a linear map $T : M_n(R) \to M_m(R)$ preserves Drazin inverses of matrices if the condition "$A \in M_n(R)$ has Drazin inverse" implies that $T(A) \in M_m(R)$ has Drazin inverse and $T(A)^D = T(A^D)$. Linear maps preserving generalized inverses of matrices are extensively studied by many authors, e.g., [1, 2, 3, 4, 6, 8, 9, 10, 11, 12].

Recall that a ring R is local if R has exactly one maximal ideal M. The ring R/M is called the residue field of R; we denote it by F. It is well known that a ring R is local if and only if for any $x \in R$, either x or $1 - x$ is invertible. Clearly, every field is a local ring. The purpose of this paper is to further explore the linear maps preserving Drazin inverses of matrices over local rings. Let $T : M_n(R) \to M_m(R)$ be a linear map preserving Drazin inverses. If $a^6 \neq 1$ for some $a \in F^*$, we prove that $T = 0$ or $n = m$ and T preserves idempotents. That is, the preserving of Drazin

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inverses can be reduced to the case of idempotents. We thereby determine the form of linear maps from $M_n(R)$ to $M_m(R)$ preserving Drazin inverses of matrices.

In what follows, $\mathbb{Z}$ and $\mathbb{Z}_n$ denote respectively the ring of integers and the ring of integers modulo n for some positive integer n. We write $J(R)$ and $U(R)$ for the Jacobson radical of R and the set of all invertible elements of R, respectively.

Throughout the paper, R is a commutative local ring with the residue field F. Also, F^* denotes the group of all nonzero elements in the field F. Moreover, E_{ij} denotes the matrices with 1 in the (i,j)-entry and 0 elsewhere, for any $i, j \in [1, n]$.

2. Main results

Let R be a local ring with the residue field F, and let T be a linear map from $M_n(R)$ to $M_m(R)$, with $m, n > 1$. The aim of this section is to investigate the linear maps preserving Drazin inverses of matrices for such a local ring R.

Lemma 2.1. *Let R be a local ring and suppose that there exists $a \in F^*$ such that $a^6 \neq 1$. Let T be a linear map from $M_n(R)$ to $M_m(R)$. If T preserves Drazin inverses of matrices and $T(E_{ii}) = 0$ for some $i \in [1, n]$, then $T = 0$.*

Proof. We claim that $|F| \geq 4$. If not, $|F^*| < 3$, and so either $x^3 = 1$ or $x^3 = -1$ for any $x \in F^*$. This shows that $x^6 = 1$, a contradiction. We may assume $T(E_{11}) = 0$. Since $|F| \geq 4$, we find some $\overline{a} \notin \{\overline{0}, \overline{1}, \overline{-1}\}$. Then $a \in U(R)$. Let $x \in U(R)$. Then $(E_{11} + xE_{1j})^D = E_{11} + xE_{1j}$ for any $j \in [2, n]$. Hence,

$$T(E_{11} + xE_{1j})^3 = T(E_{11} + xE_{1j}),$$

and so $x^3T(E_{1j})^3 = xT(E_{1j})$. In particular choose $x = 1$ and $x = a$; we see that

$$T(E_{1j})^3 = T(E_{1j}), \quad a^2T(E_{1j})^3 = T(E_{1j}).$$

Thus, we have $(a-1)(a+1)T(E_{1j}) = 0$. Clearly, $a-1$, $a+1 \in U(R)$; hence, $T(E_{1j}) = 0$. Likewise, $T(E_{j1}) = 0$ for any $j \in [2, n]$. Set $X = E_{11} + E_{1j} + E_{j1}$ and $A = E_{1j} + E_{j1} - E_{jj}$, where $j \in [2, n]$. Then $T(X) = 0$ and $A^D = X$. Then $T(A)^k = T(A)^{k+1}T(X) = 0$, and so $(-1)^kT(E_{jj}) = 0$. Hence $T(E_{jj}) = 0$. Therefore $T = 0$, as asserted. □

Lemma 2.2. *Let R be a local ring and suppose that there exists $a \in F^*$ such that $a^6 \neq 1$. Let T be a linear map from $M_n(R)$ to $M_m(R)$. If T preserves Drazin inverses of matrices, then*

$$T(E_{ii})T(E_{jj}) = T(E_{jj})T(E_{ii}) = 0$$

for any distinct $i, j \in [1, n]$.

Proof. Since $a^6 \neq 1$ for some $a \in F^*$, from the above discussion we easily see that $|F| \geq 4$, and so we can find some $\overline{a} \notin \{\overline{0}, \overline{1}, \overline{-1}\}$. Then $a \in U(R)$. Let $A = E_{ii} + a^{-1}E_{jj}$, $X = E_{ii} + aE_{jj}$ for distinct $i, j \in [1, n]$. Then $X = A^D$. Hence we have

$$T(E_{ii} + aE_{jj})T(E_{ii} + a^{-1}E_{jj}) = T(E_{ii} + a^{-1}E_{jj})T(E_{ii} + aE_{jj}).$$

It follows that

$$(a^{-1} - a)(T(E_{ii})T(E_{jj}) - T(E_{jj})T(E_{ii})) = 0.$$

As $a^2 - 1 \in U(R)$, we see that $T(E_{ii})T(E_{jj}) = T(E_{jj})T(E_{ii})$.

Let $i, j \in [1, n]$ be distinct. By hypothesis, there exists some $a \in U(R)$ such that $\overline{a}^6 \neq \overline{1}$ in F. Set $Y = E_{ii} + aE_{jj}$ and $B = E_{ii} + a^{-1}E_{jj}$. Then $Y = B^D$. Hence

$$T(Y) = T(B^D) = T(B)^D = (T(B)^D)^2 T(B) = T(Y)^2 T(B).$$

It follows that

$$\begin{aligned} T(E_{ii}) + aT(E_{jj}) &= \big(T(E_{ii}) + aT(E_{jj})\big)^2 \big(T(E_{ii}) + a^{-1}T(E_{jj})\big) \\ &= T(E_{ii})^3 + (a^{-1} + 2a)T(E_{ii})^2 T(E_{jj}) \\ &\quad + (2 + a^2)T(E_{ii})T(E_{jj})^2 + aT(E_{jj})^3. \end{aligned}$$

Since E_{ii} is an idempotent, $E_{ii}^D = E_{ii}$ and so $T(E_{ii})^D = T(E_{ii}^D) = T(E_{ii})$. Hence it is easy to see that

$$T(E_{ii}) = T(E_{ii})^3.$$

Similarly, $T(E_{jj}) = T(E_{jj})^3$. Hence,

$$(a^{-1} + 2a)T(E_{ii})^2 T(E_{jj}) + (2 + a^2)T(E_{ii})T(E_{jj})^2 = 0.$$

That is,

$$(1 + 2a^2)T(E_{ii})^2 T(E_{jj}) + (2a + a^3)T(E_{ii})T(E_{jj})^2 = 0. \tag{2.1}$$

Since $T(E_{ii}) = T(E_{ii})^3$ and $T(E_{jj}) = T(E_{jj})^3$, we derive

$$(2a + a^3)T(E_{ii})^2 T(E_{jj}) + (1 + 2a^2)T(E_{ii})T(E_{jj})^2 = 0. \tag{2.2}$$

Combining (2.1) and (2.2), we have

$$(a - 1)(a^2 - a + 1)\big(T(E_{ii})^2 T(E_{jj}) - T(E_{ii})T(E_{jj})^2\big) = 0.$$

Clearly,

$$\overline{(a-1)(a^2 - a + 1)(a + 1)(a^2 + a + 1)} = \overline{a^6 - 1} \neq \overline{0}$$

in F; hence,

$$(a - 1)(a^2 - a + 1)(a + 1)(a^2 + a + 1) \in U(R).$$

Therefore we get

$$T(E_{ii})^2 T(E_{jj}) = T(E_{ii})T(E_{jj})^2.$$

It follows by (2.2) that

$$(a + 1)(a^2 + a + 1)T(E_{ii})^2 T(E_{jj}) = 0,$$

and then $T(E_{ii})^2 T(E_{jj}) = 0$. Consequently, we have

$$T(E_{ii})T(E_{jj}) = T(E_{ii})^3 T(E_{jj}) = 0.$$

This completes the proof. □

Lemma 2.3. *Let R be a local ring, and let $A^3 = A \in M_n(R)$. Then there exists $P \in GL_n(R)$ such that $PAP^{-1} = \operatorname{diag}(A_1, 0_{n-r})$, where $A_1^2 = I_r$ for some $r \in [0, n]$.*

Proof. Clearly, A is regular. Since R is a local ring, it follows from [5, Theorem 7.3.2] that there exist $P, Q \in GL_n(R)$ such that

$$PAQ = \text{diag}(I_r, d_1, \dots, d_{n-r}),$$

where $d_i \in J(R)$ for $i \in [1, n-r]$. Since A is regular, so is PAQ, and then each $d_i \in R$ is regular. Write $d_i = d_i x_i d_i$ for some $x_i \in R$. Then $d_i(1 - x_i d_i) = 0$. As $d_i \in J(R)$, we see that $1 - x_i d_i \in U(R)$; hence, $d_i = 0$. Therefore $PAQ = \text{diag}(I_r, 0_{n-r})$. We have

$$PAP^{-1} = \text{diag}(I_r, 0_{n-r})Q^{-1}P^{-1} = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}.$$

Since $A^3 = A$, we get

$$A_1^2(A_1, A_2) = (A_1, A_2).$$

Choose $Y = PQ$. Then we have $(A_1, A_2)Y = I_r$, and so $A_1^2 = I_r$. Hence,

$$\begin{pmatrix} I_r & A_1^{-1}A_2 \\ 0 & I_{n-r} \end{pmatrix} PAP^{-1} \begin{pmatrix} I_r & -A_1^{-1}A_2 \\ 0 & I_{n-r} \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This completes the proof. □

Lemma 2.4. *Let R be a local ring and suppose that there exists $a \in F^*$ such that $a^6 \neq 1$. Let T be a linear map from $M_n(R)$ to $M_m(R)$, $n \geq m$. If T preserves Drazin inverses of matrices, then $T = 0$ or $n = m$ and $T(I_n) = I_n$.*

Proof. If $T(E_{ii}) = 0$ for some $i \in [1, n]$, it follows by Lemma 2.1 that $T = 0$. Next, we assume that $T(E_{ii}) \neq 0$ for all $i \in [1, n]$.

Since $E_{11}^3 = E_{11}$, by virtue of Lemma 2.3 there exists $P_1 \in GL_m(R)$ such that

$$T(E_{11}) = P_1 \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} P_1^{-1},$$

where $A_1^2 = I_{r_1}$ for some $r_1 \in [0, m]$. Let

$$T(E_{22}) = P_1 \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} P_1^{-1},$$

where $X_{11} \in M_{r_1}(R)$. In view of Lemma 2.2, we have

$$T(E_{11})T(E_{22}) = T(E_{22})T(E_{11}) = 0,$$

and so $X_{12} = 0$, $X_{21} = 0$ and $X_{11} = 0$. Thus,

$$T(E_{22}) = P_1 \begin{pmatrix} 0 & 0 \\ 0 & X_{22} \end{pmatrix} P_1^{-1},$$

where $X_{22}^3 = X_{22}$. By using Lemma 2.3 again, there exists $Q_1 \in GL_{m-r_1}(R)$ such that

$$X_{22} = Q_1 \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix} Q_1^{-1},$$

where $A_2^2 = I_{r_2}$ for some $r_2 \in [0, m - r_1]$. It follows that

$$T(E_{22}) = P_1 \begin{pmatrix} I_{r_1} & 0 \\ 0 & Q_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I_{r_1} & 0 \\ 0 & Q_1^{-1} \end{pmatrix} P_1^{-1}.$$

Set $P_2 = P_1 \begin{pmatrix} I_{r_1} & 0 \\ 0 & Q_1 \end{pmatrix}$. Then

$$T(E_{22}) = P_2(0 \oplus A_2 \oplus 0)P_2^{-1}.$$

Moreover,

$$T(E_{11}) = P_2(A_1 \oplus 0 \oplus 0)P_2^{-1}.$$

By iteration of this process, we have

$$T(E_{ii}) = P(0 \oplus \cdots \oplus A_i \oplus \cdots \oplus 0)P^{-1},$$

where $A_i^2 = I_{r_i}$. Clearly, $A_i \neq 0$, and so

$$T(I_n) = T(E_{11} + \cdots + E_{nn}) = P(A_1 \oplus \cdots \oplus A_n)P^{-1},$$

where $A_i^2 = I_{r_i}$. As $n \geq m$, we see that $n = m$, and so $A_i = 1$ for any $i \in [1, n]$. Therefore $T(I_n) = I_n$, as asserted. □

Lemma 2.5. *Let R be a local ring and suppose that there exists $a \in F^*$ such that $a^6 \neq 1$. Let T be a linear map from $M_n(R)$ to $M_n(R)$. If T preserves Drazin inverses of matrices, then $T(E_{ii}) = T(E_{ii})^2$ for all $i \in [1, n]$.*

Proof. If $T = 0$, then the result holds. We may assume that $T \neq 0$. In light of Lemma 2.4, $T(I_n) = I_n$. As in the proof of Lemma 2.1, since $|F| \geq 4$, we can find some $\overline{x} \notin \{\overline{0}, \overline{1}, \overline{2}\}$. Clearly, we have

$$(I_n + (x^{-1} - 1)E_{ii})^D = I_n + (x - 1)E_{ii},$$

and so

$$T(I_n + (x-1)E_{ii})T(I_n + (x^{-1} - 1)E_{ii})T(I_n + (x-1)E_{ii}) = T(I_n + (x-1)E_{ii}).$$

Set $A = T(E_{ii})$. Then,

$$(I_n + (x-1)A)(I_n + (x^{-1} - 1)A)(I_n + (x-1)A) = I_n + (x-1)A.$$

Hence,

$$\begin{aligned}(I_n + (x-1)A)(I_n(x-1+1) - (x-1)A)(I_n + (x-1)A) \\ = I_n(x-1+1) + (x-1+1)(x-1)A.\end{aligned}$$

This shows that

$$\begin{aligned}(I_n + (x-1)A)(I_n + (x-1)(I_n - A))(I_n + (x-1)A) \\ = I_n + (x-1)(I_n + A) + (x-1)^2 A,\end{aligned}$$

and so

$$(x-1)^2(A - A^2)(I_n + (x-1)A) = 0.$$

Since $A^3 = A$, we have

$$(x-1)^2(2-x)(A-A^2) = 0.$$

As $x-1,\ x-2 \in U(R)$, we see that $A^2 = A$, as asserted. □

We have accumulated all the information necessary to prove the following result.

Theorem 2.6. *Let R be a local ring and suppose that there exists $a \in F^*$ such that $a^6 \neq 1$. Let T be a linear map from $M_n(R)$ to $M_m(R)$, $n \geq m$. If T preserves Drazin inverses of matrices, then $T = 0$ or $n = m$ and T preserves idempotents.*

Proof. Suppose that $T \neq 0$. In view of Lemma 2.4, $n = m$ and $T(I_n) = I_n$. Let $M^2 = M \in M_n(R)$. Then there exists $Q \in GL_n(R)$ such that $M = Q(I_r \oplus 0)Q^{-1}$. Let $T_1(X) = T(QXQ^{-1})$. Then T_1 is a linear map from $M_n(R)$ to $M_n(R)$ and it preserves Drazin inverses of matrices with $T_1(I_n) = I_n$. By Lemma 2.2,

$$T_1(E_{ii})T_1(E_{jj}) = T_1(E_{jj})T_1(E_{ii}) = 0$$

for any distinct $i, j \in [1, n]$. By Lemma 2.5, $T_1(E_{ii}) = T_1(E_{ii})^2$ for any $i \in [1, n]$. Therefore,

$$\begin{aligned} T(M) &= T(Q(I_r \oplus 0)Q^{-1}) \\ &= T_1(I_r \oplus 0) \\ &= \sum_{i=1}^{r} T_1(E_{ii}) \\ &= \sum_{i=1}^{r} T_1(E_{ii})^2 \\ &= \left(\sum_{i=1}^{r} T_1(E_{ii})\right)^2 \\ &= (T_1(I_r \oplus 0))^2 \\ &= T(M)^2, \end{aligned}$$

as asserted. □

Note that the trivial map $T = 0$ preserves Drazin inverses of matrices. For the nonzero case, we have the following.

Corollary 2.7. *Let R be a local ring with $2, 3, 7 \in U(R)$, and let T be a nonzero linear map from $M_n(R)$ to $M_m(R)$. Then T preserves Drazin inverses of matrices if and only if $n = m$ and either there exists $P \in GL_n(R)$ such that $T(A) = PAP^{-1}$ or there exists $P \in GL_n(R)$ such that $T(A) = PA^tP^{-1}$.*

Proof. If $a^6 = 1$ for all $x \in F^*$, then $2^6 = 1$. Hence, $3^2 \times 7 = 0$, a contradiction. Therefore we can find some $a \in F^*$ such that $a^6 \neq 1$. In view of Theorem 2.6, T preserves idempotents. Therefore we complete the proof by [3, Theorem]. □

We now construct a ring to illustrate the preceding result.

Example 2.8. Let $\mathbb{Z}_{(5)} = \{\frac{p}{q} \mid p, q \in \mathbb{Z},\ (p, q) = 1 \text{ and } 5 \nmid q\}$. Then $\mathbb{Z}_{(5)}$ is a local ring with $2, 3, 7 \in U(\mathbb{Z}_{(5)})$. Let T be a linear map from $M_n(\mathbb{Z}_{(5)})$ to $M_m(\mathbb{Z}_{(5)})$. Then T preserves Drazin inverses of matrices if and only if T has the forms as in Corollary 2.7.

Proof. Clearly, $J(\mathbb{Z}_{(5)}) = 5\mathbb{Z}_{(5)}$, and so $\mathbb{Z}_{(5)}/J(\mathbb{Z}_{(5)}) \cong 5\mathbb{Z}$. Therefore we are through by Corollary 2.7. □

The condition "$a^6 \neq 1$ for some $a \in F^*$" in Theorem 2.6 is not superfluous, as the following shows.

Example 2.9. Let $T : M_2(\mathbb{Z}_3) \to M_2(\mathbb{Z}_3)$ be the linear map given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix}.$$

Then T preserves Drazin inverses of matrices, but T does not preserve idempotents.

Proof. Clearly, $\mathbb{Z}_3$ is local. Since $T(A) = -A^t$ for any $A \in M_2(\mathbb{Z}_3)$, we easily check that T preserves Drazin inverses of matrices. But

$$T\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix},$$

which is not an idempotent. Therefore T does not preserve idempotents. Observe that in this case, $a^6 = 1$ for all $a \in \mathbb{Z}_3^*$. □

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