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ON THE MODULE INTERSECTION GRAPH OF IDEALS OF RINGS

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Abstract. Let R be a commutative ring and M an R-module. The M-intersection graph of ideals of R is an undirected simple graph, denoted by $G_M(R)$, whose vertices are non-zero proper ideals of R and two distinct vertices are adjacent if and only if $IM \cap JM \neq 0$. In this article, we focus on how certain graph theoretic parameters of $G_M(R)$ depend on the properties of both R and M. Specifically, we derive a necessary and sufficient condition for R and M such that the M-intersection graph $G_M(R)$ is either connected or complete. Also, we classify all R-modules according to the diameter value of $G_M(R)$. Further, we characterize rings R for which $G_M(R)$ is perfect or Hamiltonian or pancyclic or planar. Moreover, we show that the graph $G_M(R)$ is weakly perfect and cograph.

1. Introduction

Recently, there has been considerable attention in the literature to associating graphs with rings or modules. More specifically, there are many papers on assigning graphs to modules (see, for example, [3, 11, 13]). The present paper deals with what is known as the intersection graph. The first step in this direction was taken by Csákány and Pollák [7] in 1969. In [5], the authors introduced and studied the intersection graph of a family of non-trivial ideals of a ring. These motivated the authors of [1] to define the intersection graph of submodules of a module. In the last decade, many research articles have been published on the intersection graphs of rings and modules; for instance, see [2, 4, 12, 14]. In 2018, Heydari [10] combined the concepts of intersection graph of ideals of a ring and intersection graph of submodules of a module to define the M-intersection graph of ideals of a ring R, where M is an R-module.

Formally, let R be a commutative ring with identity and let M be an R-module. The *M-intersection graph* of ideals of R, denoted by $G_M(R)$, is a simple graph whose vertices are the non-trivial ideals of R and any two distinct vertices are adjacent if $IM \cap JM \neq 0$. Note that the properties of $G_M(R)$ depend upon M as

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well as the underlying ring R. This observation inspired the authors of the paper to investigate further properties of $G_M(R)$.

We first summarize the notations and concepts. For any $a \in M$, $\operatorname{ann}(a) = \{x \in R : ax = 0\}$ is the *annihilator ideal* of a in M. The module M is called a *faithful* R-module if $\operatorname{ann}(M) = 0$. An R-module M is a *multiplication module* if for each submodule N of M there is an ideal I of R such that $IM = N$. A module is called a *uniform module* if the intersection of any two non-zero submodules is non-zero. Throughout the paper, the notations $I(R)^*$, $\operatorname{Max}(R)$, $\operatorname{Min}(R)$, $\mathbb{Z}$ and $\mathbb{Z}_n$ denote, respectively, the set of all non-trivial ideals of R, the set of all maximal ideals of R, the set of all minimal ideals of R, the set of integers and the set of integers modulo n.

A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. On the other extreme, a *null graph* is a graph containing no edges. The complete graph and null graph on n vertices are denoted by K_n and $\overline{K_n}$, respectively. A graph G is *connected* if there is a path between any two distinct vertices of G; otherwise G is *disconnected*. Let u and v be two distinct vertices of G. If the shortest $u-v$ path is of length k, then $d(u, v) = k$. Note that $d(u, v) = \infty$ if there is no path between u and v. The *diameter* of G, denoted by $\operatorname{diam}(G)$, is the supremum of the set $\{d(u, v) : u \text{ and } v \text{ are distinct vertices of } G\}$. The *girth* of G, denoted by $\operatorname{gr}(G)$, is the length of the shortest cycle in G and $\operatorname{gr}(G) = \infty$ whenever G is a tree. An Eulerian circuit in a connected graph G is a closed trail that contains every edge of G. A connected graph that contains an Eulerian circuit is called an *Eulerian graph*. A cycle in G that contains every vertex of G is called a Hamiltonian cycle of G. A *Hamiltonian graph* is a graph that contains a Hamiltonian cycle.

By a *clique* in G we mean a complete subgraph of G, and the number of vertices in the largest clique of G is called the *clique number* of G, denoted by $\omega(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Note that, for any graph G, $\omega(G) \leq \chi(G)$. A graph G is *weakly perfect* if $\omega(G) = \chi(G)$ and G is called *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G. A graph is said to be *planar* if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. For general references on graph theory, we use Chartrand and Zhang [6].

The article is organized as follows. In Section 2, we obtain a necessary and sufficient condition for the connectedness and completeness of $G_M(R)$, followed by diameter classification of $G_M(R)$. In Section 3, we provide an equivalent condition for $G_M(R)$ to be a tree and also characterize all Noetherian rings R with unique minimal ideal for which $G_M(R)$ is perfect. Finally, in Section 4 we concentrate on the cyclic nature of $G_M(R)$. In particular, we obtain the girth value of $G_M(R)$ and discuss about Hamiltonian and pancyclic nature of $G_M(R)$.

We begin with the observation that there are R-modules for which $G_M(R)$ satisfies the extreme cases, namely null graph and complete graph. If $M = \mathbb{Z}_p \oplus Z_q$ and $R = \mathbb{Z}_{pq}$ for distinct primes p and q, then $G_M(R)$ is a null graph. Further, the following result says that $G_M(R)$ is complete whenever M is a uniform faithful R-module.

Proposition 1.1. *Let R be a commutative ring and let M be a uniform R-module. Then $G_M(R) = \overline{K_\lambda} \cup K_\beta$, where $\lambda = |A|$ and $\beta = |I(R)^* \setminus A|$, with $A = \{I \in I(R)^* : IM = 0\}$. In addition, if M is both uniform and faithful, then $G_M(R)$ is complete.*

Proof. Let $A = \{I \in I(R)^* : IM = 0\}$. If $I \in A$, then $IM \cap JM = 0$ for all $J \in I(R)^*$. This implies that A is the set of isolated vertices in $G_M(R)$. Also for any two ideals $J, K \in I(R)^* \setminus A$, we have JM and KM as non-zero submodules of M. Since M is uniform, $JM \cap KM \neq 0$. This implies that the subgraph induced by $I(R)^* \setminus A$ is complete in $G_M(R)$. Moreover, if M is faithful, then $\operatorname{ann}(M) = 0$ and so there is no ideal I in R such that $IM = 0$. Thus $A = \emptyset$. □

2. Connectedness

Connectedness is one of the significant graph theoretic properties. In this section, we investigate conditions for which the graph $G_M(R)$ is connected. The main result of this section is Theorem 2.7, in which we classify the modules according to the diameter of $G_M(R)$. In order to prove Theorem 2.7, we need a few propositions and lemmas.

Notice that if a module M is not faithful, then $\operatorname{ann}(M) \neq (0)$ and $\operatorname{ann}(M) \in I(R)^*$. Also $\operatorname{ann}(M)M = (0)$ so that $\operatorname{ann}(M)$ is an isolated vertex of $G_M(R)$. Therefore $G_M(R)$ is disconnected. In case of a faithful module, Heydari [10] provides a necessary condition for the disconnected $G_M(R)$ which is stated below.

Proposition 2.1 ([10, Theorem 1]). *Let R be a commutative ring and let M be a faithful R-module. If $G_M(R)$ is disconnected, then M is a direct sum of two R-modules.*

In the following theorem we observe a few interesting consequences of the above result. In what follows, for a given R-module M, $G(M)$ denotes the intersection graph of M (see [1]).

Theorem 2.2. *Let R be a commutative ring and let M be a faithful multiplication R-module.*

(i) *If $G_M(R)$ is connected, then every pair of maximal ideals in R have a non-trivial intersection.*
(ii) *If $|I(R)^*| \geq 2$ and $G_M(R)$ is disconnected, then either $G_M(R)$ is a null graph or any ascending (or descending) chain of ideals has exactly two non-zero ideals.*

Proof. (i) Assume that $G_M(R)$ is connected.

Claim 1: We claim that $G(M)$ is connected. Let N_1 and N_2 be two non-trivial submodules of M. Since M is a multiplication R-module, there are ideals I_1 and I_2 in R such that $I_1M = N_1$ and $I_2M = N_2$. Since $G_M(R)$ is connected, there is a path $I_1 \to I_3 \to I_4 \to \cdots \to I_\ell \to I_2$ in $G_M(R)$. This implies that $I_1M \cap I_3M \neq 0$, $I_\ell M \cap I_2M \neq 0$ and $I_jM \cap I_{j+1}M \neq 0$ for all $j = 3, \ldots, \ell - 1$.

Therefore $N_1 \to I_3M \to I_4M \to \cdots \to I_\ell M \to N_2$ is a path in $G(M)$ and so $G(M)$ is connected.

Claim 2: We claim that IM is a maximal submodule of M whenever I is a maximal ideal of R. Let I be a maximal ideal of R. Suppose N is a submodule of M such that $IM \subsetneq N$. Then there exists an ideal J in R with $JM = N$. Therefore $IM \subsetneq JM$ and so $I \subsetneq J$. This implies that $J = I$ or $J = R$. Consequently $JM = IM$ or $JM = M$. Thus the claim holds true.

Assume that I and J are maximal ideals in R. Then, by Claim 2, IM and JM are maximal submodules in M. By Claim 1, $G(M)$ is connected. So, by [1, Corollary 2.2], we have $IM \cap JM \neq 0$. We claim that $IM \cap JM = (I \cap J)M$. Since M is a multiplication R-module, by [8, Corollary 1.7], we get $IM \cap JM = (I + \operatorname{ann}(M)) \cap (J + \operatorname{ann}(M))M$. Since M is faithful, $IM \cap JM = (I \cap J)M$. Therefore $(I \cap J)M \neq 0$ so that $I \cap J \neq 0$ as M is faithful.

(ii) Assume that $|I(R)^*| \geq 2$, $G_M(R)$ is disconnected and $G_M(R)$ is not a null graph. We prove that every ideal of R is minimal. Let I be an arbitrary ideal in R. Let C_1 and C_2 be two components of $G_M(R)$. Without loss of generality, assume that $I \in C_1$. Choose an ideal $J \in C_2$. If $I + J \neq R$, then $I \to I + J \to J$ is a path in $G_M(R)$, a contradiction. Thus $I + J = R$. Since I and J are not adjacent in $G_M(R)$, $IM \cap JM = 0$. Consequently, $I \cap J = 0$. We claim that I is a minimal ideal of R. Suppose K is an ideal such that $K \subsetneq I$. Clearly $K \in C_1$. By the above argument, we get $K + J = R$ and $K \cap J = 0$. Let $x \in I$. Since $J + K = R$, $x = y + z$ for some $y \in J$ and $z \in K$. Since $y = x - z \in I$ and $I \cap J = 0$, we have $x - z = 0$ and so $x = z \in K$. Therefore $I = K$, a contradiction. Thus I is a minimal ideal in R. Equivalently any ascending (or descending) chain of ideals has exactly two non-zero ideals. □

The next result characterizes all modules for which the graph $G_M(R)$ is not connected. In this regard, notice that Chakrabarty et al. [5] characterized all disconnected intersection graphs of ideals of a ring.

Theorem 2.3. *Let R be a commutative ring and M a multiplication R-module. Then $G_M(R)$ is disconnected if and only if either M is not faithful or every non-trivial ideal of R is minimal and $|\operatorname{Min}(R)| \geq 2$.*

Proof. ($\Leftarrow$): Clearly $G_M(R)$ is disconnected whenever M is not faithful because $\operatorname{ann}(M)$ is an isolated vertex of $G_M(R)$. So assume that M is faithful. Let $I, J \in \operatorname{Min}(R)$ with $I \neq J$ and $I(R)^* = \operatorname{Min}(R)$; then $I \cap J = 0$. Since M is a multiplication R-module, by [8, Corollary 1.7], we have $IM \cap JM = (I + \operatorname{ann}(M)) \cap (J + \operatorname{ann}(M))M$. Since M is faithful, $IM \cap JM = (I \cap J)M = 0$.

If $G_M(R)$ is connected, then there is a path $I \to K_1 \to K_2 \to \cdots \to K_\ell \to J$ in $G_M(R)$, where $K_i \in I(R)^* = \operatorname{Min}(R)$ for all $i = 1, \ldots, \ell$. Since I is adjacent to K_1, $IM \cap K_1M \neq 0$ implies that $I \cap K_1 \neq 0$. Since K_1 is minimal, we have $I = K_1$. Similarly, $I = K_1 = K_2 = \cdots = K_\ell = J$, a contradiction.

($\Rightarrow$): Suppose $G_M(R)$ is disconnected. Assume M is faithful. Let I and J belong to different components, say $I \in C_1$ and $J \in C_2$, where C_1 and C_2 are two distinct components of $G_M(R)$. Then $IM \cap JM = 0$ so that $I \cap J = 0$. Clearly

$I + J = R$, for otherwise $I \to I + J \to J$ is a path in $G_M(R)$. If I is not a minimal ideal of R, then there exists $K \in I(R)^*$ such that $K \subseteq I$. Now $KM \cap IM \neq 0$ and so $K \in C_1$. Here $K + J = R$, for otherwise $I \to K + J \to J$ is a path, which is a contradiction as I and J are in different components. Let us take $i \in I$. Then $i = k + j$, where $k \in K$ and $j \in J$. Since $K \subseteq I$, we have $i - k = j \in I \cap J = (0)$. Therefore $i = k \in K$, implying that $I \subseteq K$, which leads to $I = K$. So I is minimal in R. Similarly we can prove that if L is a non-trivial ideal of R with $L \supseteq I$, then $L = I$. $\square$

Now we write the other version of the above result as follows.

Corollary 2.4. *Let R be a commutative ring and M a multiplication R-module. Then $G_M(R)$ is connected if and only if M is faithful and either $|\operatorname{Min}(R)| = 1$ or $I(R)^* \neq \operatorname{Min}(R)$.*

Next, we are interested in classifying modules M according to the diameter value of its M-intersection graph. To do it, we need to find when $\operatorname{diam}(G_M(R)) = 1$, which means $G_M(R)$ is complete. Recall that from Proposition 1.1, $G_M(R)$ is complete when M is both uniform and faithful.

Theorem 2.5. *Let R be a commutative ring and M an R-module. Then $G_M(R)$ is complete if and only if M is faithful and R is Artinian with a unique minimal ideal.*

Proof. ($\Leftarrow$): Suppose R is Artinian and M is faithful. Let I be the unique minimal ideal of R. Choose arbitrary J, K in $I(R)^*$. Since R is Artinian and I is the unique minimal ideal of R, $I \subseteq J$ and $I \subseteq K$. Therefore $JM \cap KM \neq 0$ and so J and K are adjacent in $G_M(R)$. Thus $G_M(R)$ is complete.

($\Rightarrow$): Assume $G_M(R)$ is complete. Then by Corollary 2.4, M must be faithful. Since $JM \cap KM \neq 0$ for all $J, K \in I(R)^*$, we get $J \cap K \neq 0$ for all $J, K \in I(R)^*$. This implies that $\bigcap_{I \in I(R)^*} I \neq 0$ and it is the minimal ideal of R, which is unique. It remains to prove that R is Artinian. Suppose that, on the contrary, R is not Artinian. Let I be the unique minimal ideal of R. Then there exists a chain of ideals $J_1 \supseteq J_2 \supseteq \cdots J_n \supseteq \cdots$ in R which does not contain the minimal ideal I. So $I \nsubseteq J_n$ for all $n \in \mathbb{Z}^+$. Therefore $J_n \cap I = 0$ for every n. That is, I is not adjacent to any ideal J_n in the chain and $G_M(R)$ is not complete. $\square$

The following points are worth to mention in the context of Theorem 2.5.

Remark 2.6. (a) The condition for the ring to be Artinian in Theorem 2.5 is very much required. For instance, if $R = \mathbb{Z}_4 \times \mathbb{Z}$, then R has a unique minimal ideal $I = \langle 2 \rangle \times \{0\}$. But the ideals of the form $\{0\} \times n\mathbb{Z}$ do not contain I. In such cases $I \cap J$ is zero when $J = \{0\} \times n\mathbb{Z}$.

(b) An example for the situation where $I \cap J \neq 0$ but $IM \cap JM = 0$: Consider $R = \mathbb{Z}$, $M = \mathbb{Z}_6$, $I = 2\mathbb{Z}$ and $J = 3\mathbb{Z}$; although $2\mathbb{Z} \cap 3\mathbb{Z} \neq 0$, we have $IM \cap JM = 0$, so that $G_M(R)$ is not complete.

We are now in a position to state the main theorem of this section, which classifies all R-modules M according to the diameter of $G_M(R)$.

Theorem 2.7. *Let R be a commutative ring and let M be a multiplication R-module. Then the following statements hold:*

(i) $\operatorname{diam}(G_M(R)) = \infty$ *if and only if M is not faithful or every nontrivial ideal of R is minimal and* $|\operatorname{Min}(R)| \geq 2$.

(ii) $\operatorname{diam}(G_M(R)) = 1$ *if and only if M is faithful and R is Artinian with unique minimal ideal.*

(iii) $\operatorname{diam}(G_M(R)) = 2$*, for all the remaining cases.*

Proof. (i) follows from Theorem 2.3 and (ii) follows from Theorem 2.5.

(iii) Assume $G_M(R)$ is connected and $\operatorname{diam}(G_M(R)) \neq 1$. Select I and J which are not adjacent vertices in $G_M(R)$. If $I + J \neq R$, then we have a path $I \to I + J \to J$ in $G_M(R)$ so that $\operatorname{diam}(G_M(R)) = 2$. Assume that $I + J = R$. Since $IM \cap JM = 0$, we get $M = RM = (I + J)M = IM \oplus JM$. By Theorem 2.2 (i), any two maximal ideals have non-zero intersection. Therefore, either I or J is not maximal; say I is not maximal. Now, there exists an ideal K such that $I \subset K \subset R$. Clearly $IM \cap KM \neq 0$. We claim that $JM \cap KM \neq 0$. Since $R = I + J$, for $r \in K$, $r = r' + r''$, where $r' \in I$ and $r'' \in J$. So $r'' = r - r' \in K$. Therefore $r'' \in J \cap K$. Thus $r''m \in JM \cap KM$ for $m \in M$ and so $JM \cap KM \neq 0$. Hence $I \to K \to J$ is a path in $G_M(R)$ so that $\operatorname{diam}(G_M(R)) = 2$. □

The next result gives the structure of the particular graph, namely $G_{\mathbb{Z}_n}(\mathbb{Z})$.

Theorem 2.8. *The graph $G_{\mathbb{Z}_n}(\mathbb{Z}) = \overline{K_\infty} \cup H$, where $\overline{K_\infty}$ denotes the infinite collection of isolated vertices and H is an infinite connected graph.*

Proof. Clearly $I(\mathbb{Z})^* = \{m\mathbb{Z} : m \in \mathbb{Z}\setminus\{0, 1\}\}$. Let us take $I_m = m\mathbb{Z}$. If $(m, n) = n$, then $I_m\mathbb{Z}_n = (0)$ and so I_m is an isolated vertex. Therefore there are infinite isolated vertices in $G_{\mathbb{Z}_n}(\mathbb{Z})$. If $(m, n) < n$, then $I_m\mathbb{Z}_n \neq 0$. Consider $A = \{I_m \in I(\mathbb{Z})^* : (m, n) < n\}$. Let $I_k, I_\ell \in A$ with $I_k\mathbb{Z}_n \cap I_\ell\mathbb{Z}_n = 0$. Then choose $I_t \in A$ such that $(t, n) = 1$. This implies that $I_t\mathbb{Z}_n = \mathbb{Z}_n$ so that $I_k \to I_t \to I_\ell$ is a path in $G_{\mathbb{Z}_n}(\mathbb{Z})$. Therefore the subgraph induced by A is connected. □

3. Perfectness

The theory of perfect graphs relates the concept of graph colorings to the concept of cliques. The study of perfect graphs is very significant because a number of important algorithms only work on perfect graphs and perfect graphs can be used in a wide variety of applications, ranging from scheduling to order theory to communication theory. Note that it is sufficient to prove that a graph G is perfect if and only if it does not contain an odd cycle of length greater than or equal to 5.

In this section, we investigate whether $G_M(R)$ is perfect or not. We start with the following remark on the clique number of $G_M(R)$.

Lemma 3.1. *Let R be a commutative ring and let M be a faithful R-module.*

(a) *If $\omega(G_M(R)) = 1$, then either R has a unique proper ideal or M is the direct sum of two modules. The converse holds in case M is the direct sum of two simple modules.*

(b) *If $1 < \omega(G_M(R)) < \infty$, then every chain of ideals in R is finite.*

Proof.

(a) If $\omega(G_M(R)) = 1$ and $|I(R)^*| \geq 2$, then $G_M(R)$ is not connected, and by Theorem 2.1 M is the direct sum of two modules. Conversely, suppose M is the direct sum of two simple modules. Then $IM \cap JM = 0$ for any $I, J \in I(R)^*$ and so $\omega(G_M(R)) = 1$.

(b) If a chain of ideals is infinite, then the subgraph induced by the ideals in the particular chain forms a infinite clique in $G_M(R)$, which is not possible, implying that every chain of ideals is finite. □

The next result provides several equivalent relations for which the chromatic number of $G_M(R)$ is two. The following result is a generalized version of [1, Theorem 3.4]. In what follows, the star graph is a tree consisting of one vertex adjacent to all the others and the length of a module M is the length of the longest chain of submodules of M.

Theorem 3.2. *Let M be a faithful multiplication R-module and let $|I(R)^*| \geq 2$. Then the following conditions are equivalent:*

(i) *$G_M(R)$ is a star graph.*
(ii) *$G_M(R)$ is a tree.*
(iii) *$G_M(R)$ is connected and $\chi(G_M(R)) \leq 2$.*
(iv) *R is a local ring and the length of M is less than or equal to 3. In this case, $G(M) \cong G_M(R)$.*

Proof. (i) $\Longrightarrow$ (ii): It is obvious.

(ii) $\Longrightarrow$ (iii): If $G_M(R)$ is a tree, then at most two colors are sufficient to color the graph. Further, a tree is a connected acyclic graph.

(iii) $\Longrightarrow$ (iv): Let $G_M(R)$ be connected and let $\chi(G_M(R)) \leq 2$. Let I_1 and I_2 be two maximal ideals of R. Then, by Theorem 2.2, $I_1 \cap I_2 \neq \emptyset$ and so $I_1M \cap I_2M \neq \emptyset$. Consequently $I_1M \cap (I_1 \cap I_2)M \neq \emptyset$. This implies that I_1 is adjacent to $I_1 \cap I_2$. Similarly, $I_1 \cap I_2$ is adjacent to I_2. Therefore $I_1 \to I_1 \cap I_2 \to I_2 \to I_1$ so that $G_M(R)$ contains a cycle of length 3, contradicting the fact that $\chi(G_M(R)) \leq 2$. Therefore R is local.

Let I be the unique maximal ideal of R and let J be an arbitrary ideal of R. Since every ideal is contained in I, we have $J \subset I$. If J contains an ideal K, then $K \to J \to I \to K$ is a cycle of length 3, a contradiction. Thus J is a minimal ideal of R and equivalently the length of the module is at most 3. Moreover, since M is a faithful multiplication module with finite length, we have $R \cong M$ and so $G(M) \cong G_M(R)$.

(iv) $\Longrightarrow$ (i): Suppose R is a local ring and the length of M is less than or equal to 3. Let $\mathrm{Max}(R) = \{I\}$ and $\mathrm{Min}(R) = \{I_1, \ldots, I_k\}$. Since the length of M is at most 3, we have $I(R)^* = \mathrm{Max}(R) \cup \mathrm{Min}(R)$. Clearly, $IM \cap I_iM \neq 0$ for all $i = 1, \ldots, k$. Since R is faithful, $I_iM \cap I_jM = 0$ for all $1 \leq i \neq j \leq k$. Therefore $G_M(R)$ is a star graph with internal vertex I and k leaves. □

Next, we obtain a necessary and sufficient condition for the perfectness of $G_M(R)$ when R is a direct product of Noetherian rings, each of which has a unique minimal ideal.

Theorem 3.3. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i, $1 \leq i \leq n$, is a Noetherian ring with unique minimal ideal, and let M be a faithful R-module. Then $G_M(R)$ is perfect if and only if $n \leq 4$.*

Proof. ($\Rightarrow$): Suppose $n \geq 5$. Let $I_j = (0) \times \cdots \times (0) \times R_j \times R_{j+1} \times (0) \times \cdots \times (0)$ for $j = 1, 2, 3, 4$ and $I_5 = R_1 \times (0) \times (0) \times (0) \times R_5 \times (0) \times \cdots \times (0)$. Then the subgraph induced by the set $\{I_1, I_2, I_3, I_4, I_5\}$ in $G_M(R)$ is a cycle of length 5 and so $G_M(R)$ is not perfect.

($\Leftarrow$): Assume $n \leq 4$. Note that any ideal I_k of R is of the form $I_{k_1} \times \cdots \times I_{k_n}$, where I_{k_i} is an ideal of R_i for all $i = 1, \ldots, n$. If two vertices I_k and I_ℓ are non-adjacent in $G_M(R)$, then $I_k M \cap I_\ell M = 0$. The fact that M is faithful leads to $I_k \cap I_\ell = 0$. Note that R_i is Noetherian with a unique minimal ideal for all $i = 1, \ldots, n$. Therefore if I_k is not adjacent to I_ℓ in $G_M(R)$, then either $I_{k_j} = (0)$ or $I_{\ell_j} = (0)$ for each $j = 1, \ldots, n$.

We claim that every odd cycle of length more than 4 in $G_M(R)$ must have diagonals. In order to prove the claim, suppose $I_1 \to I_2 \to I_3 \to \cdots \to I_m \to I_1$ is a cycle of odd length $m \geq 5$ in $G_M(R)$. First, let us consider the best possible choice, $n = 4$. If any three ideals from $\{I_{1_1}, I_{1_2}, I_{1_3}, I_{1_4}\}$ are the zero ideal, say $I_{1_1} = I_{1_2} = I_{1_3} = (0)$, then $I_{2_4} \neq (0)$ and $I_{m_4} \neq (0)$. So I_2 and I_m form a diagonal edge. If exactly one ideal from $\{I_{1_1}, I_{1_2}, I_{1_3}, I_{1_4}\}$ is a zero ideal, say $I_{1_1} = (0)$, then $I_{3_2} = I_{3_3} = I_{3_4} = (0)$ and $I_{4_2} = I_{4_3} = I_{4_4} = (0)$. This implies that $I_{3_1} \neq (0)$ and $I_{4_1} \neq (0)$. Since $I_3 \to I_2$ and $I_4 \to I_5$, we have $I_{2_1} \neq (0)$ and $I_{5_1} \neq (0)$. Therefore I_2 and I_5 form a diagonal edge. Thus every ideal of I_1, I_2, I_3 and I_4 can be decomposed into two zero ideals and two non-zero ideals. Let $I_{1_1} = I_{1_2} = (0)$ and $I_{1_3}, I_{1_4} \neq (0)$. Then $I_{3_3} = I_{3_4} = (0)$ and $I_{4_3} = I_{4_4} = (0)$. This implies that $I_{3_1}, I_{3_2} \neq (0)$ and $I_{4_1}, I_{4_2} \neq (0)$. Since $I_3 \to I_2$, either $I_{2_1} \neq (0)$ or $I_{2_2} \neq (0)$. So I_2 and I_4 form a diagonal edge. Therefore, the claim holds true for $n = 4$. Similar arguments as above lead us to the cases $n = 3$ and $n = 2$.

So let $n = 1$. If the length of M is less than 4, then by Theorem 3.2, $G_M(R)$ is a tree and so it is perfect. Therefore the length of M is at least 4. Since M is faithful, the subgraph induced by any chain of ideals of R is complete in $G_M(R)$. Thus $G_M(R)$ is perfect. ☐

As we have seen, the M-intersection graphs of modules are not perfect in general. So we investigate whether they are weakly perfect, and the answer is yes.

Theorem 3.4. *The graph $G_M(R)$ is weakly perfect for any R-module M.*

Proof. If $\omega(G_M(R))$ is infinite, then so is $\chi(G_M(R))$. Let $\omega(G_M(R)) = n$, a finite positive integer and let $S = \{I_1, \ldots, I_n\}$ be a clique of $G_M(R)$. Let $A = \{I \in I(R)^* : IM = 0\}$ and $A' = I(R)^* \setminus A$. Clearly the set A is independent in $G_M(R)$ and $S \subseteq A'$. If $S = A'$, then obviously $\chi(G_M(R)) = \omega(G_M(R))$. Suppose $S \subsetneq A'$. Then for every $J \in A' \setminus S$, the vertices $I_1, \ldots, I_n, J + I_1, \ldots, J + I_n$ form a clique of size $2n$, which is not possible. Therefore $J + I_1, \ldots, J + I_n$ are the same as $I_1, \ldots, I_n$ in different order. In addition, suppose there exists $J, K \in A' \setminus S$ such that J and K are adjacent in $G_M(R)$. Then $J, J + I_1, \ldots, J + I_n, K$ form a clique of size $n + 2$ in $G_M(R)$, a contradiction. Therefore the open neighborhood of each vertex in $A' \setminus S$ is in S. Since $\omega(G_M(R)) = n = |S|$, every vertex in $A' \setminus S$ is not adjacent to more than $n-1$ vertices of S. Thus $\chi(G_M(R)) = n = \omega(G_M(R))$. □

The next result determines the value of $\omega(G_M(R))$ and $\chi(G_M(R))$ when M is the direct sum of finite simple modules.

Theorem 3.5. *If M is a multiplication R-module such that $M = M_1 \oplus \cdots \oplus M_n$ where each M_i, $1 \leq i \leq n$, is simple, then*

$$\omega(G_M(R)) = \max_{1 \leq i \leq n} |\{I \in I(R)^* : IM \supseteq M_i\}|.$$

Proof. Let $V_0 = \{I \in I(R)^* : IM = (0)\}$ and $V_{n+1} = \{I \in I(R)^* : IM = M\}$. Let

$$\begin{aligned} A_1 = \{&M_1, M_1 \oplus M_2, M_1 \oplus M_3, \ldots, M_1 \oplus M_n, M_1 \oplus M_2 \oplus M_3, \ldots, \\ &M_1 \oplus M_2 \oplus M_n, M_1 \oplus M_3 \oplus M_4, \ldots, M_1 \oplus M_3 \oplus M_n, \ldots, \\ &M_1 \oplus M_{n-1} \oplus M_n, \ldots, M_1 \oplus M_2 \oplus \cdots \oplus M_{n-1}, \\ &M_1 \oplus M_3 \oplus \cdots \oplus M_n, \ldots, M_1 \oplus M_2 \oplus \cdots \oplus M_{n-2} \oplus M_n\}, \\ A_2 = \{&M_2, M_2 \oplus M_3, \ldots, M_2 \oplus M_n, M_2 \oplus M_3 \oplus M_4, \ldots, M_2 \oplus M_3 \oplus M_n, \ldots, \\ &M_2 \oplus M_3 \oplus \cdots \oplus M_n\}, \\ &\ldots \\ A_{n-1} = \{&M_{n-1}, M_{n-1} \oplus M_n\}, \text{ and} \\ A_n = \{&M_n\}. \end{aligned}$$

Define $V_1 = \{I \in I(R)^* : IM \in A_1\}$ and $V_i = \{I \in I(R)^* : IM \in A_i\}$ for $i = 2, \ldots, n$. Clearly $I(R)^* = \bigcup_{k=0}^{n+1} V_k$ and V_k's are mutually disjoint. Note that every vertex of V_0 is isolated in $G_M(R)$ and the subgraph induced by V_k for $k = 1, \ldots, n+1$ is complete in $G_M(R)$. For $1 \leq i \leq n$, let $I \in V_i$ and $J \in V_{n+1}$. Then $IM \cap JM \supseteq M_i \neq \{0\}$. Therefore every vertex in V_{n+1} is adjacent to all the vertices of V_i. Further, let $K \in V_j$ for some $1 \leq i \neq j \leq n$. Since M is a multiplication module, there exists an ideal L in R such that $LM = M_i$. Since M_i is simple, L is not adjacent to K in $G_M(R)$. Therefore, in $G_M(R)$, every vertex of $I(R)^* \setminus (V_{n+1} \cup V_i)$ is not adjacent to at least one vertex in $V_{n+1} \cup V_i$. Thus $V_{n+1} \cup V_i$ is a maximal clique in $G_M(R)$ for all $i = 1, \ldots, n$. Hence $\omega(G_M(R)) = |V_{n+1}| + \max_{1 \leq i \leq n} |V_i|$; equivalently, $\omega(G_M(R)) = \max_{1 \leq i \leq n} |\{I \in I(R)^* : IM \supseteq M_i\}|$. □

Let us close this section by applying the above result for finding the clique and coloring number of specific M-intersection graphs.

Example 3.6. Consider $M = \mathbb{Z}_{10} = \langle 2 \rangle \oplus \langle 5 \rangle$ and $R = \mathbb{Z}_{30}$. Then $I(R)^* = \{\langle 2 \rangle, \langle 3 \rangle, \langle 5 \rangle, \langle 6 \rangle, \langle 10 \rangle, \langle 15 \rangle\}$. Let $I_1 = \langle 2 \rangle$, $I_2 = \langle 3 \rangle$, $I_3 = \langle 5 \rangle$, $I_4 = \langle 6 \rangle$, $I_5 = \langle 10 \rangle$, $I_6 = \langle 15 \rangle$. Now $I_1 M = \langle 2 \rangle$, $I_2 M = M$, $I_3 M = \langle 5 \rangle$, $I_4 M = \langle 2 \rangle$, $I_5 M = \{0\}$, $I_6 M = \langle 5 \rangle$. So $V_0 = \{I_5\}$, $V_1 = \{I_1, I_4\}$, $V_2 = \{I_3, I_6\}$ and $V_3 = \{I_2\}$. Therefore by Theorem 3.5, we have $\omega(G_M(R)) = 3$. Also, by Figure 1, the chromatic number of the corresponding M-intersection graph is also 3.

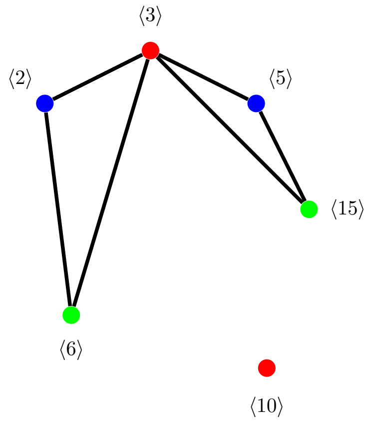

Figure 1. $G_{\mathbb{Z}_{10}}(\mathbb{Z}_{30})$

Example 3.7. Suppose $M = \mathbb{Z}_6$ and $R = \mathbb{Z}$. Then $I(R)^* = \{n\mathbb{Z} : n \in \mathbb{Z}, n \neq 0, 1\}$, $V_0 = \{6k\mathbb{Z} : 0 \neq k \in \mathbb{Z}\}$, $V_1 = \{3k\mathbb{Z} : k \in \mathbb{Z}, (2,k) = 1\}$, $V_2 = \{2k\mathbb{Z} : k \in \mathbb{Z}, (3,k) = 1\}$ and $V_3 = \{k\mathbb{Z} : k \in \mathbb{Z}, (6,k) = 1\}$. Clearly the cardinalities of V_0, V_1, V_2 and V_3 are infinite and so $\omega(G_M(R)) = \chi(G_M(R)) = \infty$.

4. Cyclic subgraph and planarity

In this section, we discuss about some cyclic substructure and planarity of $G_M(R)$. Let us start with the girth value of $G_M(R)$. In [10, Theorem 4], Heydari determined the girth of $G_M(R)$ in case of a multiplication R-module M. We now generalize it.

Theorem 4.1. *Let M be an R-module. If $G_M(R)$ contains a cycle, then* $\mathrm{gr}(G_M(R)) = 3$. *That is,* $\mathrm{gr}(G_M(R)) \in \{3, \infty\}$.

Proof. Let $B = \{I \in I(R)^* : IM = 0\}$ and $B' = I(R)^* \setminus B$. Clearly every vertex in B is isolated in $G_M(R)$. Suppose $\mathrm{gr}(G_M(R)) \geq 4$. Then there are ideals $I_1, I_2, I_3, I_4 \in B'$ such that $I_1 \to I_2 \to I_3 \to I_4$ is a path in $G_M(R)$. Suppose there are ideals I_k and I_ℓ, $1 \leq k \neq \ell \leq 4$, such that I_k and I_ℓ are not comparable. Then $I_k \to I_k + I_\ell \to I_k + I_m \to I_k$, where $m \in \{1,2,3,4\} \setminus \{k, \ell\}$, is a cycle of length 3, a contradiction. Therefore the ideals I_1, I_2, I_3 and I_4 are comparable. Then we can compile into two cases. If $I_1 \subseteq I_2$, $I_3 \subseteq I_2$ and $I_3 \subseteq I_4$, then $I_3 \subseteq I_2 \cap I_4$. So $(I_2 \cap I_4)M \neq 0$, which implies that $I_2 \to I_3 \to I_4 \to I_2$ is a cycle of length 3. If $I_2 \subseteq I_1$ and $I_2 \subseteq I_3$, then $I_2 \subseteq I_1 \cap I_3$ and so $I_1 \to I_2 \to I_3 \to I_1$ is a cycle of length 3. Hence $\mathrm{gr}(G_M(R)) = 3$. □

Now, the proof of Theorem 4.1 together with the application of Theorem 3.2 leads to the following result. Observe that the proof of Theorem 4.1 says that the path on 4 vertices is not an induced subgraph of $G_M(R)$ whenever $G_M(R)$ contains a cycle. Further, Theorem 3.2 says that $G_M(R)$ is a star graph whenever $G_M(R)$

does not contain any cycle. Recall that a graph G is called *cograph* if P_4, the path on 4 vertices, is not an induced subgraph of G.

Theorem 4.2. *The graph $G_M(R)$ is a cograph for any faithful R-module M.*

The next result helps us to find the length of the largest induced cycle and induced path in $G_M(R)$. The length of the longest induced path in G is called the *induced detour number* of G, denoted by $\mathrm{idn}(G)$, and the maximum length of an induced cycle in G is called the *induced circumference* of G, denoted by icir(G); see [9].

Theorem 4.3. *Let R be the direct product of n local rings and let M be a faithful R-module. Then $G_M(R)$ contains a cycle C_k for every $k \in \{3, 4, \ldots, n\}$. Further,* $\mathrm{idn}(G_M(R)) = n - 1$ *and* $\mathrm{icir}(G_M(R)) = n$.

Proof. Let $R = R_1 \times R_2 \times \cdots \times R_n$, where each R_i, $1 \leq i \leq n$, is local. Let $I_j = (0) \times \cdots \times (0) \times R_j \times R_{j+1} \times (0) \times \cdots \times (0)$ for $j = 1, \ldots, n-1$ and $I_n = R_1 \times (0) \times \cdots \times (0) \times R_n$. For a fixed k, $3 \leq k \leq n$, take $J_k = R_1 \times (0) \times \cdots \times (0) \times R_k \times (0) \times \cdots \times (0)$. Now the subgraph induced by the vertices $\{I_1, \ldots, I_{k-1}, J_k\}$ forms a cycle of length k in $C_M(R)$. So C_n is an induced subgraph of $G_M(R)$. Further, a similar proof technique as in Theorem 3.3 leads us to say that every cycle of length more than n in $G_M(R)$ must have a diagonal. Thus $\mathrm{icir}(G_M(R)) = n$. Also, in this case, it is equivalent to saying that $\mathrm{idn}(G_M(R)) = n - 1$. □

Other important concepts related with the cyclic structure of a graph are those of being Eulerian or Hamiltonian. First of all note that if $R \neq I + J$ for any $I, J \in I(R)^*$, then we can have a cycle $I_1 \to I_1 + I_2 \to I_2 + I_3 \to \cdots \to I_{n-1} + I_n \to I_n + I_1 \to I_1$ of length $n+1$ in $G_M(R)$, where $I_1, \ldots, I_n \in I(R)^*$. In this regard, we add a couple of observations.

Remark 4.4. Let M be a faithful R-module and $|I(R)^*| \geq 3$.

(a) If R is an Artinian local ring or M is uniform, then $G_M(R)$ is complete and so it is Hamiltonian.
(b) If M is not a faithful R-module, then $\mathrm{ann}(M)$ is an isolated vertex in $G_M(R)$, so that $G_M(R)$ is not Hamiltonian.

Proposition 4.5. *Let M be a uniform R-module. Then $G_M(R)$ is Eulerian if and only if $|\{I \in I(R)^* : IM \neq \{0\}\}|$ is odd.*

Proof. If $I \in I(R)^*$ is such that $IM = \{0\}$, then I is isolated and so no edge is incident with I. Since M is uniform, the subgraph induced by $\{I \in I(R)^* : IM \neq \{0\}\}$ in $G_M(R)$ is complete. Now, the result follows from the fact that the complete graph of n vertices is Eulerian if and only if n is odd. □

Next we characterize all commutative Noetherian rings according to the Hamiltonian nature of $G_M(R)$ under the assumption that M is faithful and R is the direct product of rings which has a unique minimal ideal. At this point notice that if M is not faithful, then by Theorem 2.3, $G_M(R)$ is disconnected. Also, if R is decomposed into more than two rings, then $I(R)^* \neq \mathrm{Min}(R)$.

Theorem 4.6. *Let R be a Noetherian ring with $|I(R)^*| \geq 3$ and let M be a faithful R-module. Let $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i, $1 \leq i \leq n$, is a local ring with a unique minimal ideal. Then $G_M(R)$ is Hamiltonian if and only if $R \ncong R_1 \times R_2$ where either R_1 or R_2 is a field.*

Proof. ($\Rightarrow$): Assume that $G_M(R)$ is Hamiltonian. Let I_i denote an arbitrary ideal of R_i for all $i = 1, \ldots, n$. Choose $A_j = \{(0) \times \cdots \times (0) \times I_j \times I_{j+1} \times \cdots \times I_n : I_j \neq (0)\}$. Clearly $I(R)^* = \bigcup_{j=1}^{n} A_j$. Since each R_i is Noetherian with a unique minimal ideal, we have two vertices $J = J_1 \times \cdots \times J_n$ and $K = K_1 \times \cdots \times K_n$ adjacent in $G_M(R)$ if and only if $J_i \neq (0)$ and $K_i \neq (0)$ for some $i \in \{1, \ldots, n\}$. Therefore the subgraph induced by the set A_j is complete in $G_M(R)$.

Suppose $n \geq 3$. Now, for every $j \in \{1, \ldots, n-2\}$, fix the vertices $u_j = (0) \times \cdots \times (0) \times K_j \times K_{j+1} \times K_{n-2} \times (0) \times K_n \in A_j$ and $v_j = (0) \times \cdots \times (0) \times K_j \times K_{j+1} \times \cdots \times K_{n-1} \times K_n \in A_j$ with $K_{n-1} \neq (0)$. Note that $u_j \neq v_j$ for all $j = 1, \ldots, n-2$ and the subgraph induced by the set $\{u_1, \ldots, u_{n-2}, v_1, \ldots, v_{n-2}\}$ in $G_M(R)$ is complete. Now, in $G_M(R)$, start a path P from the vertex $u_1 \in A_1$ and travel along all the vertices of A_1 and end up in $v_1 \in A_1$. Since v_1 is adjacent to $u_2 \in A_2$, continue the path P to u_2, then travel all the vertices of A_2 and end up in $v_2 \in A_2$. Continuing this process $n-2$ steps, we get the path P containing all the vertices of $\bigcup_{j=1}^{n-2} A_j$ which end up in v_{n-2}. Notice that the vertex v_{n-2} is adjacent to all the vertices of A_{n-1} in $G_M(R)$. So now extend the path P to any vertex of A_{n-1} and then travel along all the vertices of A_{n-1} and arrive at the vertex $(0) \times \cdots \times (0) \times K_{n-1} \times R_n \in A_{n-1}$. Since the vertex $(0) \times \cdots \times (0) \times K_{n-1} \times R_n$ is adjacent to all the vertices of A_n, by repeating the process to the vertices of A_n, we get P as a Hamiltonian path which ends at the vertex $(0) \times \cdots \times (0) \times R_n \in A_n$. Now the edge between $(0) \times \cdots \times (0) \times K_{n-1} \times R_n$ and $(0) \times \cdots \times (0) \times R_n$ leads to a Hamiltonian cycle in $G_M(R)$.

Suppose $n = 2$. There are three possibilities: (1) both R_1 and R_2 are not fields, (2) one of R_1 or R_2 is a field and the other is not a field, or (3) both R_1 and R_2 are fields.

Case 1: Assume both R_1 and R_2 are not fields. Let $K_1 \in I(R_1)^*$ and $K_2 \in I(R_2)^*$. Here start a path Q from $K_1 \times R_2 \in A_1$, travel along all the vertices of A_1 and end at $K_1 \times K_2 \in A_1$. Then move the path Q to the vertex $(0) \times R_2 \in A_2$ and move on to all the vertices of A_2 and finally end at $(0) \times K_2 \in A_1$. Now the path Q together with the edge between $(0) \times K_2$ and $K_1 \times R_2$ forms a Hamiltonian cycle in $G_M(R)$.

Case 2: Assume that R_1 is a field and R_2 is not a field. If $K_{2_1}, K_{2_2} \in I(R_2)^*$ with $K_{2_1} \neq K_{2_2}$, then start a path Q from $R_1 \times K_{2_1} \in A_1$, travel along all the vertices of A_1 and end at $R_1 \times K_{2_2} \in A_1$. Then move into A_2 through the vertex $(0) \times K_{2_1} \in A_2$ and travel all the vertices of A_2 and end at up $(0) \times R_2 \in A_1$. Now the path Q together with the edge between $(0) \times R_2$ and $R_1 \times K_{2_1}$ serves as a Hamiltonian cycle in $G_M(R)$. If $I(R_2)^* = \{K_2\}$, then $I(R)^* = \{R_1 \times (0), R_1 \times K_2, (0) \times K_2, (0) \times R_2\}$. Therefore the vertex $R_1 \times (0)$ is an end vertex in $G_M(R)$ so that $G_M(R)$ does not contain a cycle. Thus, in this case, $|I(R_2)^*| \neq 1$.

Case 3: If R_1 and R_2 are fields, then $|I(R)^*| = 2$, a contradiction.

Suppose $n = 1$. Since $|I(R)^*| \geq 3$, we have that $G_M(R)$ is complete and so is Hamiltonian.

($\Leftarrow$): This part is obvious. $\square$

Next we look into the pancyclic nature of $G_M(R)$. Recall that a graph G of order $m \geq 3$ is pancyclic if G contains cycles of all lengths from 3 to m. Let $n \geq 3$ and let us denote C as the Hamiltonian cycle identified in the above theorem. Now remove the vertices one by one from C. Let us start with the vertices A_n. First remove the internal vertices of A_n and at the last remove the vertex $(0) \times \cdots \times (0) \times R_n \in A_n$. Since the subgraph induced by A_n is complete in $G_M(R)$, we get a cycle every time when we remove a vertex from A_n. Since the subgraph induced by $\{u_1, \ldots, u_{n-2}, v_1, \ldots, v_{n-2}\}$ is complete in $G_M(R)$, we can pass on from one A_j to another A_k for $1 \leq j \neq k \leq n-2$. Similarly we can remove the vertices one by one for all A_j's from $j = n-2$ to $j = 1$ and still we get a cycle in every step of vertex removal. Thus we get cycles of all lengths as subgraphs of $G_M(R)$. It is not hard to verify the same for the cases $n = 2$ and $n = 1$. Hence $G_M(R)$ is Hamiltonian if and only if $G_M(R)$ is pancyclic. So the following statement holds true.

Theorem 4.7. *Let R be an Artinian ring, $|I(R)^*| \geq 3$ and let M be a faithful R-module. Then $G_M(R)$ is pancyclic if and only if $R \not\cong \mathbb{F} \times S$, where $\mathbb{F}$ is a field and S is a local ring with unique proper ideal.*

We close the paper with the planarity properties of $G_M(R)$. The first result in this regard completely characterizes the planar M-intersection graph in the case of R having exactly one minimal ideal.

Lemma 4.8. *Let M be a faithful R-module. If R contains a unique minimal ideal, then $G_M(R)$ is planar if and only if $|I(R)^*| \leq 4$.*

Proof. ($\Rightarrow$): Assume that $G_M(R)$ is a planar graph. Since R has a unique minimal ideal, say I, we have that I must be contained in every proper ideal of R. Suppose $|I(R)^*| > 4$. Let $I, J_1, J_2, J_3, J_4 \in I(R)^*$. Note that $J_k \cap J_\ell \supseteq I$ for all $1 \leq k \neq \ell \leq 4$ so that $(J_k M \cap J_l M) \supseteq IM \neq 0$. Therefore I, J_1, J_2, J_3, J_4 form a complete subgraph of $G_M(R)$ and so K_5 is a subgraph of $G_M(R)$, a contradiction. Thus $|I(R)^*| \leq 4$.

($\Leftarrow$): This part follows from the fact that every graph of order less than or equal to four is planar. $\square$

Note that, in the above result, IM (or J_1M, J_2M, J_3M, J_4M) may be zero if M is not faithful.

Next we are going to characterize all rings for which the graph $G_M(R)$ is planar, where R is the direct product of local rings. Before that, we prove a lemma.

Lemma 4.9. *Let R be the direct product of n local rings and let M be a faithful R-module. Then K_{2^n} is a subgraph of $G_M(R)$.*

Proof. Let $R = R_1 \times \cdots \times R_n$, where each R_i is local with maximal ideal $\mathfrak{m}_i$ for $i = 1, \ldots, n$. Let $A = \{I_1 \times \cdots \times I_n : I_i = R_i \text{ or } \mathfrak{m}_i \text{ for } i = 1, \ldots, n\} \setminus \{R_1 \times \cdots \times R_n\}$. Then $A \subset I(R)^*$ and $|A| = 2^n - 1$. Note that the subgraph induced by A is complete and the vertex $(0) \times R_2 \times \cdots \times R_n$ is adjacent to all the vertices of A in $G_M(R)$. Thus K_{2^n} is a subgraph of $G_M(R)$. □

To end this section, recall that every Artinian ring can be decomposed into local rings. So the following result is valid for all Artinian rings.

Theorem 4.10. *Let $R = R_1 \times \cdots \times R_n$, where each $R_i, 1 \leq i \leq n$, is local and let M be a faithful R-module. Then $G_M(R)$ is planar if and only if either $n = 2$ with R_1 being a field and R_2 having at most one proper ideal, or $n = 1$ with R_1 having at most 4 proper ideals.*

Proof. ($\Rightarrow$): Assume that $G_M(R)$ is planar. If $n \geq 3$, then by Lemma 4.9, K_8 is a subgraph of $G_M(R)$, a contradiction.

Suppose $n = 2$. There are three possibilities:

Case 1: Assume both R_1 and R_2 are not fields. Let $K_1 \in I(R_1)^*$ and $K_2 \in I(R_2)^*$. Then the subgraph induced by the vertex subset $\{K_1 \times (0), K_1 \times K_2, K_1 \times R_2, R_1 \times (0), R_1 \times K_2\}$ is K_5, a contradiction.

Case 2: Assume that R_1 is a field and R_2 is not a field. If $K_{2_1}, K_{2_2} \in I(R_2)^*$ with $K_{2_1} \neq K_{2_2}$, then the subgraph induced by the set $\{R_1 \times K_{2_1}, R_1 \times K_{2_2}, (0) \times K_{2_1}, (0) \times K_{2_2}, (0) \times R_2\}$ is K_5, a contradiction. If $|I(R_2)^*| = 1$, then $|I(R)^*| \leq 4$. So $G_M(R)$ is planar.

Case 3: If R_1 and R_2 are fields, then $|I(R)^*| = 2$ so that $G_M(R)$ is planar.

Suppose $n = 1$. If $|I(R)^*| \geq 5$, then K_5 is a subgraph of $G_M(R)$, a contradiction. Thus $|I(R)^*| \leq 4$.

($\Leftarrow$): This part is trivial. □

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References

[1] S. Akbari, H. A. Tavallaee and S. K. Ghezelahmad, Intersection graph of submodules of a module, *J. Algebra Appl.* **11** (2012), no. 1, 1250019, 8 pp. MR 2900889.

[2] D. F. Anderson and A. Badawi, The generalized total graph of a commutative ring, *J. Algebra Appl.* **12** (2013), no. 5, 1250212, 18 pp. MR 3055569.

[3] H. Ansari-Toroghy, S. Habibi and M. Hezarjaribi, On the graph of modules over commutative rings II, *Filomat* **32** (2018), no. 10, 3657–3665. MR 3901512.

[4] T. Asir and T. Tamizh Chelvam, The intersection graph of gamma sets in the total graph of a commutative ring—II, *J. Algebra Appl.* **12** (2013), no. 4, 1250199, 14 pp. MR 3037273.

[5] I. Chakrabarty, S. Ghosh, T. K. Mukherjee and M. K. Sen, Intersection graphs of ideals of rings, *Discrete Math.* **309** (2009), no. 17, 5381–5392. MR 2548554.

[6] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, Tata McGraw-Hill, India, 2006.

[7] B. Csákány and G. Pollák, The graph of subgroups of a finite group, *Czechoslovak Math. J.* **19(94)** (1969), 241–247. MR 0249328.

[8] Z. A. El-Bast and P. F. Smith, Multiplication modules, *Comm. Algebra* **16** (1988), no. 4, 755–779. MR 0932633.

[9] *Graph Theory—Favorite Conjectures and Open Problems. 2*, edited by R. Gera, T. W. Haynes and S. T. Hedetniemi, Problem Books in Mathematics, Springer, Cham, 2018. MR3890100.

[10] F. Heydari, The M-intersection graph of ideals of a commutative ring, *Discrete Math. Algorithms Appl.* **10** (2018), no. 3, 1850038, 11 pp. MR 3809554.

[11] P. Malakooti Rad, Planar torsion graph of modules, *Filomat* **30** (2016), no. 2, 367–372. MR 3497916.

[12] J. Matczuk, M. Nowakowska and E. R. Puczyłowski, Intersection graphs of modules and rings, *J. Algebra Appl.* **17** (2018), no. 7, 1850131, 20 pp. MR 3813704.

[13] S. B. Pejman, Sh. Payrovi and S. Babaei, The intersection graph of annihilator submodules of a module, *Opuscula Math.* **39** (2019), no. 4, 577–588. MR 3964763.

[14] T. Tamizh Chelvam and T. Asir, The intersection graph of gamma sets in the total graph of a commutative ring—I, *J. Algebra Appl.* **12** (2013), no. 4, 1250198, 18 pp. MR 3037272.

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