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APPROXIMATION VIA STATISTICAL K_a^2-CONVERGENCE ON TWO-DIMENSIONAL WEIGHTED SPACES

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ABSTRACT. We give a non-regular statistical summability method named statistical K_a^2-convergence and prove a Korovkin type approximation theorem for this new and interesting convergence method on two-dimensional weighted spaces. We also study the rate of statistical K_a^2-convergence by using the weighted modulus of continuity and afterwards we present a non-trivial application.

1. Introduction and preliminaries

Statistical convergence of a number sequence for single sequences was first introduced, independently, by Fast and Steinhaus ([9, 19]). Then this notion was extended to double sequences by Moricz ([14]).

A double sequence $x = (x_{ij})$ is said to be convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $J = J(\varepsilon) \in \mathbb{N}$, the set of all natural numbers, such that $|x_{ij} - L| < \varepsilon$ whenever $i, j > J$, where L is called the Pringsheim limit of x and denoted by $P\text{-}\lim_{i,j} x_{ij} = L$ (see [18]). If there exists a positive number c such that $|x_{ij}| \leq c$ for all $(i,j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then a double sequence is called bounded. As it is well known, a convergent single sequence is bounded whereas a convergent double sequence need not be bounded.

If $S \subset \mathbb{N}^2$ is a two-dimensional subset of positive integers, the double natural density of S is given by

$$d_2(S) := P\text{-}\lim_{i,j} \frac{|\{(m,n) \in S : m \leq i,\, n \leq j\}|}{ij}, \quad \text{if it exists,}$$

where $|S|$ denotes the cardinality of S. The number sequence $x = (x_{ij})$ is statistically convergent to L provided that for every $\varepsilon > 0$, the set

$$S := S_{ij}(\varepsilon) := \{m \leq i,\, n \leq j : |x_{mn} - L| \geq \varepsilon\}$$

has natural density zero; in that case we write $st_2\text{-}\lim_{i,j} x_{ij} = L$. As it is well known, a double sequence that is convergent in Pringsheim's sense is statistically convergent to the same value, but the converse is not always true and a statistically

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convergent double sequence need not be bounded. Also, a well-known relationship states that statistical convergence for double sequences and almost convergence for double sequences ([15]) overlap, and they are important in convergence theory. There is also a new and interesting convergence method named K_a-convergence. First, Lazic and Jovovic defined the K_a-convergence for single sequences in 1993 ([12]). Then, more recently, Yıldız ([21]) has extended this notion to double sequences. She has showed that statistical convergence, almost convergence and K_a-convergence for double sequences overlap. Our goal in this paper is to give a non-regular (not necessarily positive) statistical summability method named statistical K_a^2-convergence and prove a Korovkin type approximation theorem for this convergence method on two-dimensional weighted spaces. We study the rate of statistical K_a^2-convergence by using the weighted modulus of continuity. Finally, we present an application that shows that our result is stronger than proven by earlier authors.

We first begin to recall the notion of K_a^2-convergence ([21]). This new convergence method is associated to the four-dimensional matrix

$$\left(\begin{array}{cccc} \begin{pmatrix} a_{11} & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix} & \begin{pmatrix} a_{12} & a_{11} & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix} & \begin{pmatrix} a_{13} & a_{12} & a_{11} & 0 & . \\ 0 & 0 & 0 & 0 & . \\ . & . & . & . & . \end{pmatrix} & \cdots \\ \begin{pmatrix} a_{12} & 0 & 0 & . \\ a_{11} & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix} & \begin{pmatrix} a_{22} & a_{21} & 0 & . \\ a_{12} & a_{11} & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix} & \begin{pmatrix} a_{23} & a_{22} & a_{21} & 0 & . \\ a_{13} & a_{12} & a_{11} & 0 & . \\ 0 & 0 & 0 & 0 & . \\ . & . & . & . & . \end{pmatrix} & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{array}\right).$$

Let $a = (a_{ij})$ and $x = (x_{ij})$ be double sequences. Set $K_a^2(x) = y$, where $y = (y_{ij})$ and $y_{ij} = \sum_{m=1}^{i} \sum_{n=1}^{j} a_{i-m+1 j-n+1} x_{mn}$ $(i, j = 1, 2, 3, \dots)$. Then it is said that $y = (y_{ij})$ is the K_a^2-*transformation* of the double sequence $x = (x_{ij})$.

Definition 1.1 ([21]). The double sequence of real numbers $x = (x_{ij})$ is said to be K_a^2-*convergent* to the number L if its K_a^2-transformation $y = (y_{ij})$ converges to the number L in Pringsheim's sense, i.e. $P\text{-}\lim_{i,j} y_{ij} = L$, and we denote this limit by $K_a^2\text{-}\lim_{i,j} x_{ij} = L$.

For a double sequence $a = (a_{ij})$ we introduce the following two conditions:

- $P\text{-}\lim\limits_{i,j} \sum\limits_{m=1}^{i} \sum\limits_{n=1}^{j} |a_{mn}|$ exists. (1.1)
- There exists a positive integer c such that $\sum\limits_{(i,j)\in\mathbb{N}^2} |a_{ij}| < c$. (1.2)

Proposition 1.2 ([21]). *Let $a = (a_{ij})$ be a double sequence.*

(i) *If $x = (x_{ij})$ is convergent in Pringsheim's sense, with $P\text{-}\lim_{i,j} x_{ij} = L$, and conditions* (1.1) *and* (1.2) *are satisfied, then*

$$K_a^2\text{-}\lim_{i,j} x_{ij} = L \sum_{(i,j)\in\mathbb{N}^2} a_{ij}.$$

(ii) *A convergence method K_a^2 is RH-regular if and only if conditions* (1.1) *and* (1.2) *and the relation*

$$\sum_{(i,j)\in\mathbb{N}^2} a_{ij} = 1$$

are satisfied.

Now, we give the following definition of statistical K_a^2-convergence.

Definition 1.3. The double sequence $x = (x_{ij})$ of real numbers is *statistically K_a^2-convergent* to the number L if its K_a^2-transformation $y = (y_{ij})$ statistically converges to the number L, i.e. $st_2\text{-}\lim_{i,j} y_{ij} = L$, and we write $(st_2)K_a^2\text{-}\lim_{i,j} x_{ij} = L$.

Example 1.4. Let

$$a = (a_{ij}) = \begin{pmatrix} 1 & 0 & 0 & . \\ 0 & 1 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix}$$

and

$$x = (x_{ij}) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 1 & 1 & 1 & 1 & 1 & . \\ 1 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & . \\ 1 & 0 & \frac{1}{3} & \frac{2}{3} & 1 & 1 & 1 & . \\ 1 & 0 & 1 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & . \\ 1 & 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & 1 & . \\ . & . & . & . & . & . & . & . \end{pmatrix}. \tag{1.3}$$

It is easy to check that

$$\left(\sum_{m=1}^{i}\sum_{n=1}^{j} a_{i-m+1j-n+1}x_{ij} - 1\right) = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} & 0 & . \\ 0 & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix}.$$

Then, clearly, $x = (x_{ij})$ is K_a^2-convergent to the number 1, hence $x = (x_{ij})$ is statistically K_a^2-convergent to the number 1, i.e.

$$(st_2)K_a^2\text{-}\lim_{i,j} x_{ij} = 1.$$

2. Approximation via statistical K_a^2-convergence

Korovkin type theorems have a very important role in approximation theory. Many mathematicians have investigated and improved them ([6, 7, 16, 17, 20, 22]). The convergence of a sequence of positive linear operators defined on weighted space was first studied by Gadjiev ([10]). These results were later improved by Duman and Orhan via statistical convergence ([8]) and by Atlıhan and Orhan via summability methods ([2]). Recently, Cao and Liu ([5]) studied Korovkin type theorems for two variable functions by means of a single sequence on weighted spaces, and more recently Akdag ([1]) studied this theorem for double sequences.

Now, we turn our attention to the two-dimensional weighted spaces.

A real valued function ρ is called a *weight function* if it is continuous on $\mathbb{R}^2$ and for all $(x,y)\in\mathbb{R}^2$,

$$\rho(x,y)\geq 1 \quad\text{and}\quad \lim_{\sqrt{x^2+y^2}\to\infty}\rho(x,y)=\infty. \tag{2.1}$$

Let B_ρ denote the weighted space of real valued functions f defined on $\mathbb{R}^2$ and satisfying $|f(x,y)|\leq M_f\rho(x,y)$ (for all $x,y\in\mathbb{R}$), where M_f is a constant depending on the function f. The weighted subspace C_ρ of B_ρ is given by

$$C_\rho:=\left\{f\in B_\rho : f \text{ is continuous on } \mathbb{R}^2\right\}.$$

The spaces B_ρ and C_ρ are Banach spaces with the norm (see [5])

$$\|f\|_\rho:=\sup_{(x,y)\in\mathbb{R}^2}\frac{|f(x,y)|}{\rho(x,y)}.$$

Let ρ_1 and ρ_2 be two weight functions satisfying (2.1). Assume also that the condition

$$\lim_{\sqrt{x^2+y^2}\to\infty}\frac{\rho_1(x,y)}{\rho_2(x,y)}=0 \tag{2.2}$$

holds. If T is a positive linear operator from C_{ρ_1} into B_{ρ_2}, then we know that

$$\|T\|_{C_{\rho_1}\to B_{\rho_2}}:=\|T(\rho_1)\|_{\rho_2}.$$

Now we recall the following Korovkin type approximation theorems on two-dimensional weighted space for double sequences, before giving our main theorem.

Throughout the paper we use the test functions F_r $(r=0,1,2,3)$ defined by

$$F_0(x,y)=\frac{\rho_1(x,y)}{1+x^2+y^2},\qquad F_1(x,y)=\frac{x\rho_1(x,y)}{1+x^2+y^2},$$
$$F_2(x,y)=\frac{y\rho_1(x,y)}{1+x^2+y^2},\qquad F_3(x,y)=\frac{\left(x^2+y^2\right)\rho_1(x,y)}{1+x^2+y^2}.$$

Theorem 2.1 ([1]). *Assume that the functions ρ_1 and ρ_2 are weight functions satisfying* (2.2) *and let (L_{ij}) be a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then, for all $f \in C_{\rho_1}$,*

$$P\text{-}\lim_{i,j} \|L_{ij}(f) - f\|_{\rho_2} = 0$$

if

$$P\text{-}\lim_{i,j} \|L_{ij}(F_r) - F_r\|_{\rho_1} = 0, \quad r = 0, 1, 2, 3.$$

Theorem 2.2 ([1]). *Let ρ_1 and ρ_2 be weight functions satisfying* (2.2). *Assume that (L_{ij}) is a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then, for all $f \in C_{\rho_1}$,*

$$st_2\text{-}\lim_{i,j} \|L_{ij}(f) - f\|_{\rho_2} = 0$$

if

$$st_2\text{-}\lim_{i,j} \|L_{ij}(F_r) - F_r\|_{\rho_1} = 0, \quad r = 0, 1, 2, 3.$$

Theorem 2.3 ([1]). *Let (L_{ij}) be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}, where ρ_1 and ρ_2 satisfy condition* (2.2). *Then, for all $f \in C_{\rho_1}$,*

$$P\text{-}\lim_{i,j} \|D_{k,l,i,j}(f) - f\|_{\rho_2} = 0 \quad \textit{uniformly in } k, l,$$

where $D_{k,l,i,j}(f) = \frac{1}{ij} \sum_{m=k}^{k+i-1} \sum_{n=l}^{l+j-1} L_{mn}(f; x, y)$ provided that

$$P\text{-}\lim_{i,j} \|D_{k,l,i,j}(F_r) - F_r\|_{\rho_1} = 0 \quad \textit{uniformly in } k, l \ (r = 0, 1, 2, 3).$$

Now, we give a Korovkin type approximation theorem for statistical K_a^2-convergence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Our proofs take into consideration the revised proofs presented in [3].

First of all, we give the following remark.

Let (L_{ij}) be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Suppose that $a = (a_{ij})$ is a double sequence and conditions (1.1) and (1.2) are satisfied. Then

$$\|T_{ij}\|_{C_{\rho_1} \to B_{\rho_2}} := \|T_{ij}(\rho_1)\|_{\rho_2} \leq \sum_{m=1}^{i} \sum_{n=1}^{j} |a_{i-m+1j-n+1}| \, \|L_{mn}(\rho_1)\|_{\rho_2},$$

where $T_{ij}(f; x, y) = \sum_{m=1}^{i} \sum_{n=1}^{j} a_{i-m+1j-n+1} L_{mn}(f; x, y)$. Since $\rho_1 \in C_{\rho_1}$, we have $L_{ij}(\rho_1) \in B_{\rho_2}$ and therefore $\|L_{ij}(\rho_1)\|_{\rho_2} < \infty$. Also, since conditions (1.1) and (1.2) are satisfied, we have $\|T_{ij}\|_{C_{\rho_1} \to B_{\rho_2}} < \infty$, which implies the uniform boundedness of T_{ij} from C_{ρ_1} into B_{ρ_2}.

Now we present the next lemma, as we need to prove our main theorem.

Lemma 2.4. *Let $a = (a_{ij})$ be a double sequence such that conditions* (1.1) *and* (1.2) *are satisfied. Assume that (L_{ij}) is a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}, where ρ_1 and ρ_2 are weight functions satisfying condition* (2.2). *If*

$$st_2\text{-}\lim_{i,j} \left\|T_{ij}^*(F_r) - F_r\right\|_{\rho_1} = 0, \tag{2.3}$$

for $r = 0, 1, 2, 3$, *where* $T^*_{ij}(f; x, y) = \sum_{m=1}^{i} \sum_{n=1}^{j} |a_{i-m+1j-n+1}| L_{mn}(f; x, y)$, *then, for any* $s > 0$ *and for all* $f \in C_{\rho_1}$, *we have*

$$st_2\text{-}\lim_{i,j} \left(\sup_{\sqrt{x^2+y^2} \leq s} \frac{|T_{ij}(f; x, y) - f(x, y)|}{\rho_2(x, y)} \right) = 0,$$

where $T_{ij}(f; x, y) = \sum_{m=1}^{i} \sum_{n=1}^{j} a_{i-m+1j-n+1} L_{mn}(f; x, y)$.

Proof. Let $f \in C_{\rho_1}$ and $\sqrt{x^2 + y^2} \leq s$. Since f is continuous on $\mathbb{R}^2$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(s, t) - f(x, y)| < \varepsilon$ with $|s - x| < \delta$ and $|t - y| < \delta$. When $|s - x| \geq \delta$ or $|t - y| \geq \delta$, we have

$$\begin{aligned}
&|f(s, t) - f(x, y)| \\
&\qquad < 2M_f \rho_1(x, y) \rho_1(s, t) \\
&\qquad = 2M_f \rho_1(x, y) F_0(s, t) \left(1 + s^2 + t^2\right) \\
&\qquad \leq 4M_f \rho_1(x, y) F_0(s, t) \left(1 + x^2 + y^2 + (s - x)^2 + (t - y)^2\right) \\
&\qquad = 4M_f \rho_1(x, y) F_0(s, t) \left[(s - x)^2 + (t - y)^2\right] \left(\frac{1 + x^2 + y^2}{(s - x)^2 + (t - y)^2} + 1 \right) \\
&\qquad \leq K_{\rho_1}(x, y) \left[(s - x)^2 + (t - y)^2\right] F_0(s, t),
\end{aligned}$$

where $K_{\rho_1}(x, y) := 4M_f \rho_1(x, y) \left\{1 + \frac{1+x^2+y^2}{\delta^2}\right\}$. So, for all $(s, t) \in \mathbb{R}^2$ and $\sqrt{x^2 + y^2} \leq s$, we see that

$$|f(s, t) - f(x, y)| < \varepsilon + K_{\rho_1}(x, y) \left[(s - x)^2 + (t - y)^2\right] F_0(s, t). \tag{2.4}$$

Then, we can write

$$\begin{aligned}
&|T_{ij}(f; x, y) - f(x, y)| \\
&\qquad = \left| \sum_{m=1}^{i} \sum_{n=1}^{j} a_{i-m+1j-n+1} L_{mn}(f; x, y) - f(x, y) \right| \\
&\qquad \leq \sum_{m=1}^{i} \sum_{n=1}^{j} |a_{i-m+1j-n+1}| L_{mn}(|f(s, t) - f(x, y)|; x, y) \\
&\qquad\qquad + |f(x, y)| \left| \sum_{m=1}^{i} \sum_{n=1}^{j} |a_{i-m+1j-n+1}| L_{mn}(1; x, y) - 1 \right| \\
&\qquad = T^*_{ij}(|f(s, t) - f(x, y)|; x, y) + |f(x, y)| \left|T^*_{ij}(1; x, y) - 1\right| \\
&\qquad \leq T^*_{ij}(\varepsilon + K_{\rho_1}(x, y) \left[(s - x)^2 + (t - y)^2\right] F_0(s, t); x, y) \\
&\qquad\qquad + |f(x, y)| \left|T^*_{ij}(1; x, y) - 1\right| \\
&\qquad = \varepsilon T^*_{ij}(1; x, y) + K_{\rho_1}(x, y) T^*_{ij}\left(F_0(s, t) \left[(s - x)^2 + (t - y)^2\right]; x, y\right) \\
&\qquad\qquad + |f(x, y)| \left|T^*_{ij}(1; x, y) - 1\right|.
\end{aligned}$$

Hence

$$\begin{aligned}\sup_{\sqrt{x^2+y^2}\le s} &\left|T_{ij}(f;x,y)-f(x,y)\right| \\ &\le \varepsilon H_1 \sup_{\sqrt{x^2+y^2}\le s} \frac{T_{ij}^*(1;x,y)}{\rho_1(x,y)} \\ &\quad + H_2 \sup_{\sqrt{x^2+y^2}\le s} T_{ij}^*\big(F_0(s,t)\big[(s-x)^2+(t-y)^2\big];x,y\big) \\ &\quad + H_3 \sup_{\sqrt{x^2+y^2}\le s} \left|T_{ij}^*(1;x,y)-1\right|,\end{aligned} \tag{2.5}$$

where

$$\begin{aligned} H_1 &:= H_1(s) := \sup_{\sqrt{x^2+y^2}\le s} \rho_1(x,y), \\ H_2 &:= H_2(s) := \sup_{\sqrt{x^2+y^2}\le s} K_{\rho_1}(x,y), \\ H_3 &:= H_3(s) := \sup_{\sqrt{x^2+y^2}\le s} \left|f(x,y)\right|. \end{aligned}$$

For any $s \in \mathbb{R}$, we have

$$\begin{aligned}\sup_{\sqrt{x^2+y^2}\le s} & T_{ij}^*(F_0(s,t)\big[(s-x)^2+(t-y)^2\big];x,y) \\ &= \sup_{\sqrt{x^2+y^2}\le s} \sum_{m=1}^{i}\sum_{n=1}^{j} \left|a_{i-m+1j-n+1}\right| \big[L_{mn}\left(\left(s^2+t^2\right)F_0(s,t);x,y\right) \\ &\quad - 2xL_{mn}\left(sF_0(s,t);x,y\right) - 2yL_{mn}\left(tF_0(s,t);x,y\right) \\ &\quad + \left(x^2+y^2\right)L_{mn}\left(F_0(s,t);x,y\right)\big] \\ &\le \sup_{\sqrt{x^2+y^2}\le s} \Big\{\left|T_{ij}^*(F_3;x,y)-F_3(x,y)\right| + 2\left|x\right|\left|T_{ij}^*(F_1;x,y)-F_1(x,y)\right| \\ &\quad + 2\left|y\right|\left|T_{ij}^*(F_2;x,y)-F_2(x,y)\right| + \left(x^2+y^2\right)\left|T_{ij}^*(F_0;x,y)-F_0(x,y)\right|\Big\}, \\ &\le H_4\left\{\sup_{\sqrt{x^2+y^2}\le s} \frac{\left|T_{ij}^*(F_0;x,y)-F_0(x,y)\right|}{\rho_1(x,y)} + \sup_{\sqrt{x^2+y^2}\le s} \frac{\left|T_{ij}^*(F_1;x,y)-F_1(x,y)\right|}{\rho_1(x,y)}\right. \\ &\quad \left. + \sup_{\sqrt{x^2+y^2}\le s} \frac{\left|T_{ij}^*(F_2;x,y)-F_2(x,y)\right|}{\rho_1(x,y)} + \sup_{\sqrt{x^2+y^2}\le s} \frac{\left|T_{ij}^*(F_3;x,y)-F_3(x,y)\right|}{\rho_1(x,y)}\right\},\end{aligned} \tag{2.6}$$

where

$$H_4 \coloneqq H_4(s) \coloneqq \max\left\{ \sup_{\sqrt{x^2+y^2}\le s} \rho_1(x,y), 2 \sup_{\sqrt{x^2+y^2}\le s} |x|\, \rho_1(x,y), \right.$$
$$\left. 2 \sup_{\sqrt{x^2+y^2}\le s} |y|\, \rho_1(x,y), \sup_{\sqrt{x^2+y^2}\le s} \left(x^2+y^2\right) \rho_1(x,y) \right\}.$$

Since $F_0 \in C_{\rho_1}$ and

$$F_0(x,y)|T^*_{ij}(1;x,y)-1| \le T^*_{ij}(|F_0(s,t)-F_0(x,y)|;x,y) + |T^*_{ij}(F_0;x,y)-F_0(x,y)|,$$

it follows from (2.4) that

$$\left|T^*_{ij}(1;x,y)-1\right| \le \frac{1}{F_0(x,y)}\Big\{\varepsilon T^*_{ij}(1;x,y) + \left|T^*_{ij}(F_0;x,y)-F_0(x,y)\right|$$
$$+ K_{\rho_1}(x,y) T^*_{ij}(F_0(s,t)\left[(s-x)^2+(t-y)^2\right];x,y)\Big\}.$$

Hence, we have, for any $s \in \mathbb{R}$ and for all $i,j \in \mathbb{N}$, that

$$\begin{aligned} \sup_{\sqrt{x^2+y^2}\le s} &\left|T^*_{ij}(1;x,y)-1\right| \\ &\le H_5 \left\{ \sup_{\sqrt{x^2+y^2}\le s} \frac{\left|T^*_{ij}(F_0;x,y)-F_0(x,y)\right|}{\rho_1(x,y)} + \varepsilon \sup_{\sqrt{x^2+y^2}\le s} \frac{T^*_{ij}(1;x,y)}{\rho_1(x,y)} \right\} \\ &\quad + H_6 \sup_{\sqrt{x^2+y^2}\le s} T^*_{ij}(F_0(s,t)\left[(s-x)^2+(t-y)^2\right];x,y), \end{aligned} \tag{2.7}$$

where

$$H_5 \coloneqq H_5(s) \coloneqq \sup_{\sqrt{x^2+y^2}\le s} \frac{\rho_1(x,y)}{F_0(x,y)},$$
$$H_6 \coloneqq H_6(s) \coloneqq \sup_{\sqrt{x^2+y^2}\le s} \frac{K_{\rho_1}(x,y)}{F_0(x,y)}.$$

Also, by (2.3), for each $r = 0,1,2,3$, there exists a set $K_r \subseteq \mathbb{N}^2$ such that $d_2(K_r) = 1$ and $P\text{-}\lim_{(i,j)\in K_r} \left\|T^*_{ij}(F_r)-F_r\right\|_{\rho_1} = 0$, i.e., given $\varepsilon > 0$ there exists $J_r(\varepsilon)$ such that for all $(i,j) \in K_r$ and $i,j \ge J_r(\varepsilon)$ we have $\left\|T^*_{ij}(F_r)-F_r\right\|_{\rho_1} < \varepsilon$. Hence, there is a positive number A_r such that $\left\|T^*_{ij}(F_r)-F_r\right\|_{\rho_1} \le A_r$ for every $(i,j) \in K_r$. Let

$K := \bigcap_{r=0}^{3} K_r$. Observe that $d_2(K) = 1$. So, for every $(i,j) \in K$, we have

$$\begin{aligned}\sup_{\sqrt{x^2+y^2}\leq s} \frac{T_{ij}^*(1;x,y)}{\rho_1(x,y)} &\leq \sup_{\sqrt{x^2+y^2}\leq s} \frac{T_{ij}^*(\rho_1;x,y)}{\rho_1(x,y)} \\ &\leq \sup_{\sqrt{x^2+y^2}\leq s} \frac{\left|T_{ij}^*(\rho_1;x,y) - \rho_1(x,y)\right|}{\rho_1(x,y)} + 1 \\ &\leq \sup_{\sqrt{x^2+y^2}\leq s} \frac{\left|T_{ij}^*(F_3;x,y) - F_3(x,y)\right|}{\rho_1(x,y)} \\ &\quad + \sup_{\sqrt{x^2+y^2}\leq s} \frac{\left|T_{ij}^*(F_0;x,y) - F_0(x,y)\right|}{\rho_1(x,y)} + 1 \leq A,\end{aligned}$$

where $A := A_3 + A_1 + 1$. From which, for every $(i,j) \in K$,

$$\sup_{\sqrt{x^2+y^2}\leq s} \frac{T_{ij}^*(1;x,y)}{\rho_1(x,y)} < \infty \tag{2.8}$$

follows and considering the inequalities (2.5), (2.6) and (2.7), we have

$$\begin{aligned}&\sup_{\sqrt{x^2+y^2}\leq s} \left| \sum_{m=1}^{i} \sum_{n=1}^{j} a_{i-m+1j-n+1} L_{mn}(f;x,y) - f(x,y) \right| \\ &\leq H \Bigg\{ \varepsilon \sup_{\sqrt{x^2+y^2}\leq s} \frac{T_{ij}^*(1;x,y)}{\rho_1(x,y)} + \sup_{\sqrt{x^2+y^2}\leq s} \frac{\left|T_{ij}^*(F_0;x,y) - F_0(x,y)\right|}{\rho_1(x,y)} \\ &\quad + \sup_{\sqrt{x^2+y^2}\leq s} \frac{\left|T_{ij}^*(F_1;x,y) - F_1(x,y)\right|}{\rho_1(x,y)} \\ &\quad + \sup_{\sqrt{x^2+y^2}\leq s} \frac{\left|T_{ij}^*(F_2;x,y) - F_2(x,y)\right|}{\rho_1(x,y)} \\ &\quad + \sup_{\sqrt{x^2+y^2}\leq s} \frac{\left|T_{ij}^*(F_3;x,y) - F_3(x,y)\right|}{\rho_1(x,y)} \Bigg\},\end{aligned}$$

where $H := \max\{H_1 + H_3H_5, H_4(H_2 + H_3H_6) + H_3H_5\}$. By using (2.8) and taking

$$M = \max\left\{ H \sup_{\sqrt{x^2+y^2}\leq s} \frac{T_{ij}^*(1;x,y)}{\rho_1(x,y)}, H \right\},$$

we get

$$\sup_{\sqrt{x^2+y^2}\leq s} \frac{\left|T_{ij}(f;x,y) - f(x,y)\right|}{\rho_2(x,y)} \leq M \left\{ \varepsilon + \sum_{r=0}^{3} \sup_{\sqrt{x^2+y^2}\leq s} \frac{\left|T_{ij}^*(F_r;x,y) - F_r(x,y)\right|}{\rho_1(x,y)} \right\} \tag{2.9}$$

for all $(i,j) \in K$ for some $M > 0$ independent of (x,y). Now, for a given $\eta > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \frac{\eta}{M}$. Then, define

$$S(\eta) := \left\{ (i,j) \in K : \sup_{\sqrt{x^2+y^2} \leq s} \frac{|T_{ij}(f;x,y) - f(x,y)|}{\rho_2(x,y)} \geq \eta \right\}$$

and, for $r = 0,1,2,3$,

$$S_r(\eta) := \left\{ (i,j) \in K : \sup_{\sqrt{x^2+y^2} \leq s} \frac{\left|T^*_{ij}(F_r;x,y) - F_r(x,y)\right|}{\rho_1(x,y)} \geq \frac{\frac{\eta}{M} - \varepsilon}{3} \right\}.$$

It follows from (2.9) that $S(\eta) \subset \bigcup_{r=0}^{3} S_r(\eta)$ and hence $d_2(S(\eta)) \leq \sum_{r=0}^{3} d_2(S_r(\eta))$. Then using the hypothesis (2.3) we get the desired result. ☐

Now, we are ready to give our main Korovkin type approximation theorem.

Theorem 2.5. *Let $a = (a_{ij})$, ρ_1 and ρ_2 be as in Lemma 2.4, and assume that conditions* (1.1) *and* (1.2) *hold. Let (L_{ij}) be a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then for all $f \in C_{\rho_1}$,*

$$(st_2)\, K_a^2\text{-}\lim_{i,j} \|L_{ij}(f) - f\|_{\rho_2} = 0,$$

i.e.

$$st_2\text{-}\lim_{i,j} \|T_{ij}(f) - f\|_{\rho_2} = 0,$$

where $T_{ij}(f;x,y) = \sum_{m=1}^{i} \sum_{n=1}^{j} a_{i-m+1j-n+1} L_{mn}(f;x,y)$, *provided that*

$$st_2\text{-}\lim_{i,j} \left\|T^*_{ij}(F_r) - F_r\right\|_{\rho_1} = 0, \quad r = 0,1,2,3, \tag{2.10}$$

where $T^*_{ij}(f;x,y) = \sum_{m=1}^{i} \sum_{n=1}^{j} |a_{i-m+1j-n+1}|\, L_{mn}(f;x,y)$.

Proof. Let K be as in the proof of Lemma 2.4. Observe that the hypothesis (2.10) implies that, for all $(i,j) \in K$, $T^*_{ij}(F_r;x,y) - F_r(x,y) \in B_{\rho_1}$ and hence $T^*_{ij}(F_r;x,y) \in B_{\rho_1}$ for $r = 0,1,2,3$. Since $\rho_1 = F_0 + F_3$, we also get $T^*_{ij}(\rho_1) \in B_{\rho_1}$ for each $(i,j) \in K$. Hence if $f \in C_{\rho_1}$ then we obtain $T^*_{ij}(f) \in B_{\rho_1}$. Furthermore, since

$$\begin{aligned} \left\|T^*_{ij}\right\|_{C_{\rho_1} \to B_{\rho_1}} &= \left\|T^*_{ij}(\rho_1)\right\|_{\rho_1} \\ &= \sup_{(x,y) \in \mathbb{R}^2} \frac{\left|T^*_{ij}(\rho_1;x,y)\right|}{\rho_1(;x,y)} \leq M_1 < \infty, \end{aligned}$$

we get

$$\begin{aligned}\|T_{ij}\|_{C_{\rho_1}\to B_{\rho_1}} = \|T_{ij}(\rho_1)\|_{\rho_1} &= \sup_{(x,y)\in\mathbb{R}^2} \frac{|T_{ij}(\rho_1;x,y)|}{\rho_1(x,y)} \\ &\le \sup_{(x,y)\in\mathbb{R}^2} \frac{T^*_{ij}(\rho_1;x,y)}{\rho_1(x,y)} \le M_1 < \infty.\end{aligned}$$

Therefore we may write for a given $f \in C_{\rho_1}$ that

$$\|T_{ij}(f)\|_{\rho_1} \le \|T_{ij}\|_{C_{\rho_1}\to B_{\rho_1}} \|f\|_{\rho_1} \le M_1 \|f\|_{\rho_1}. \tag{2.11}$$

Now for a given $\varepsilon > 0$, pick an $s_0 > 0$ such that $\frac{\rho_1(x,y)}{\rho_2(x,y)} \le \varepsilon$ for every $\sqrt{x^2+y^2} \ge s_0$ This is possible by (2.2), and hence we may write for $f \in C_{\rho_1}$:

$$\begin{aligned}\|T_{ij}(f) - f\|_{\rho_2} &= \sup_{(x,y)\in\mathbb{R}^2} \frac{|T_{ij}(f;x,y) - f(x,y)|}{\rho_2(x,y)} \\ &\le \sup_{\sqrt{x^2+y^2}\le s_0} \frac{|T_{ij}(f;x,y) - f(x,y)|}{\rho_2(x,y)} \\ &\quad + \sup_{\sqrt{x^2+y^2}\ge s_0} \frac{|T_{ij}(f;x,y) - f(x,y)|}{\rho_2(x,y)} \frac{\rho_1(x,y)}{\rho_1(x,y)} \\ &\le \sup_{\sqrt{x^2+y^2}\le s_0} \frac{|T_{ij}(f;x,y) - f(x,y)|}{\rho_2(x,y)} + \varepsilon \|T_{ij}(f) - f\|_{\rho_1} \\ &\le \sup_{\sqrt{x^2+y^2}\le s_0} \frac{|T_{ij}(f;x,y) - f(x,y)|}{\rho_2(x,y)} + \varepsilon\big(\|T_{ij}(f)\|_{\rho_1} + \|f\|_{\rho_1}\big),\end{aligned}$$

and by (2.11) we immediately get, for all $(i,j) \in K$,

$$\|T_{ij}(f) - f\|_{\rho_2} \le \sup_{\sqrt{x^2+y^2}\le s_0} \frac{|T_{ij}(f;x,y) - f(x,y)|}{\rho_2(x,y)} + \varepsilon \|f\|_{\rho_1}(M_1+1).$$

Hence, using Lemma 2.4, the proof is completed. □

Using the Pringsheim limit instead of the statistical limit, we can get the following result, which is a Korovkin type theorem for K_a^2-convergence.

Corollary 2.6. *Let $a = (a_{ij})$, ρ_1 and ρ_2 be as in Lemma 2.4, and assume that conditions (1.1) and (1.2) hold. Let (L_{ij}) be a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then for all $f \in C_{\rho_1}$,*

$$K_a^2\text{-}\lim_{i,j} \|L_{ij}(f) - f\|_{\rho_2} = 0,$$

i.e.

$$P\text{-}\lim_{i,j} \|T_{ij}(f) - f\|_{\rho_2} = 0,$$

provided that

$$P\text{-}\lim_{i,j} \left\|T^*_{ij}(F_r) - F_r\right\|_{\rho_1} = 0, \quad r = 0,1,2,3.$$

3. Rate of convergence

In this section, we obtain the rate of statistical K_a^2-convergence. Now, defining the weight function ρ_1 in Theorem 2.5 by $\rho_1(x,y) = 1 + x^2 + y^2$ on $\mathbb{R}^2$, we study the rate of statistical K_a^2-convergence by using the following weighted modulus of continuity:

$$w_{\rho_1}(f,\delta) = \sup_{\sqrt{(s-x)^2+(t-y)^2}\leq\delta} \frac{|f(s,t) - f(x,y)|}{\rho_1(s,t) + \rho_1(x,y)},$$

where δ is a positive constant and $f \in C_{\rho_1}$. It can be easily seen that, for any $c > 0$ and all $f \in C_{\rho_1}$,

$$w_{\rho_1}(f,c\delta) \leq (2 + \lfloor c \rfloor)\, w_{\rho_1}(f,\delta),$$

where $\lfloor c \rfloor$ is defined as the greatest integer less than or equal to c. Also, following [3], we may write, for any $\delta > 0$:

$$|f(s,t) - f(x,y)| \leq 4\rho_1(s,t)\rho_1(x,y)\left(1 + \frac{(s-x)^2 + (t-y)^2}{\delta^2}\right) w_{\rho_1}(f,\delta).$$

If we use the same operators L_{mn} as in Theorem 2.5, we can write, for any $\delta > 0$:

$$\begin{aligned}
&|T_{ij}(f;x,y) - f(x,y)| \\
&\quad = \left| \sum_{m=1}^{i}\sum_{n=1}^{j} a_{i-m+1j-n+1} L_{mn}(f;x,y) - f(x,y) \right| \\
&\quad \leq \sum_{m=1}^{i}\sum_{n=1}^{j} |a_{i-m+1j-n+1}|\, L_{mn}(|f(s,t) - f(x,y)|;x,y) \\
&\qquad\quad + |f(x,y)| \left| \sum_{m=1}^{i}\sum_{n=1}^{j} |a_{i-m+1j-n+1}|\, L_{mn}(F_0;x,y) - F_0(x,y) \right| \\
&\quad = T_{ij}^{*}(|f(s,t) - f(x,y)|;x,y) + |f(x,y)| \left| T_{ij}^{*}(F_0;x,y) - F_0(x,y) \right| \\
&\quad \leq 4\rho_1(x,y) w_{\rho_1}(f,\delta)\, T_{ij}^{*}\left(\rho_1(s,t) + \frac{\rho_1(s,t)\left[(s-x)^2 + (t-y)^2\right]}{\delta^2};x,y \right) \\
&\qquad\quad + |f(x,y)| \left| T_{ij}^{*}(F_0;x,y) - F_0(x,y) \right| \\
&\quad \leq 4\rho_1(x,y) w_{\rho_1}(f,\delta) \\
&\qquad\quad \times \left\{ \left| T_{ij}^{*}(\rho_1;x,y) - \rho_1(x,y) \right| + \rho_1(x,y) + \frac{1}{\delta^2} T_{ij}^{*}\left(\rho_1 \varphi_{(x,y)};x,y\right) \right\} \\
&\qquad\quad + |f(x,y)| \left| T_{ij}^{*}(F_0;x,y) - F_0(x,y) \right|,
\end{aligned}$$

where $\varphi_{(x,y)}(s,t) := (s-x)^2 + (t-y)^2$, and we obtain

$$\begin{aligned}\|T_{ij}f-f\|_{\rho_2^2} \leq 4\|\rho_1\|_{\rho_2} w_{\rho_1}(f,\delta) \\ \times\left\{\left\|T^*_{ij}(\rho_1)-\rho_1\right\|_{\rho_2}+\|\rho_1\|_{\rho_2}+\frac{1}{\delta^2}\left\|T^*_{ij}\left(\rho_1\varphi_{(x,y)}\right)\right\|_{\rho_2}\right\} \\ +\|f\|_{\rho_2}\|\rho_1\|_{\rho_2}\left\|T^*_{ij}(F_0)-F_0\right\|_{\rho_1},\end{aligned} \tag{3.1}$$

provided that $T^*_{ij}\left(\rho_1\varphi_{(x,y)}\right)\in B_{\rho_2}$.

Theorem 3.1. *Let $a=(a_{ij})$ and (L_{ij}) be the same as in Theorem 2.5. Let $T^*_{ij}\left(\rho_1\varphi_{(x,y)}\right)\in B_{\rho_2}$, where $\varphi_{(x,y)}(s,t):=(s-x)^2+(t-y)^2$. If*

(i) $st_2\text{-}\lim_{i,j}\left\|T^*_{ij}(F_0)-F_0\right\|_{\rho_1}=0$,
(ii) $st_2\text{-}\lim_{i,j}\left\|T^*_{ij}(\rho_1)-\rho_1\right\|_{\rho_2}=0$,
(iii) $st_2\text{-}\lim_{i,j} w_{\rho_1}(f,\delta)=0$, *where* $\delta:=\sqrt{\left\|T^*_{ij}(\varphi_{(x,y)})\right\|_{\rho_2}}$,

then for all $f\in C_{\rho_1}$,

$$st_2\text{-}\lim_{i,j}\|T_{ij}(f)-f\|_{\rho_2^2}=0.$$

Proof. By (3.1) and (i), (ii), (iii), we get the desired result. □

4. Application

We now present an example of a double sequence of positive linear operators that satisfies the conditions of Theorem 2.5 but does not satisfy the conditions of Theorem 2.1, Theorem 2.2 and Theorem 2.3.

Example 4.1. Let us consider the following linear positive operators given in [11], defined by:

$$L_{ij}(f;x,y):=\sum_{v=0}^{\infty}\sum_{\mu=0}^{\infty} f\left(\frac{v}{\beta_i},\frac{\mu}{\gamma_j}\right)K_{i,v}(x)K_{j,\mu}(y)\frac{(-\alpha_i)^v}{v!}\frac{(-\alpha_j)^\mu}{\mu!}.$$

Here (α_i), (β_i) and (γ_i) are real number sequences satisfying:

(a) $\lim_{i\to\infty}\beta_i=\infty$ and $\lim_{j\to\infty}\gamma_j=\infty$,
(b) $\lim_{i\to\infty}\frac{\alpha_i}{\beta_i}=0$ and $\lim_{j\to\infty}\frac{\alpha_j}{\gamma_j}=0$,
(c) $\lim_{i\to\infty} i\frac{\alpha_i}{\beta_i}=1$ and $\lim_{j\to\infty} j\frac{\alpha_j}{\gamma_j}=1$,

and $K_{i,v}(x)$ and $K_{j,\mu}(y)$ are functions satisfying:

(i) For any natural $i,j,v,\mu=0,1,2,\dots$ and for any $x,y\in[0,\infty)$,

$$(-1)^v K_{i,v}(x)\geq 0 \quad\text{and}\quad (-1)^\mu K_{j,\mu}(y)\geq 0,$$

(ii) For any $x,y\in[0,\infty)$,

$$\sum_{v=0}^{\infty}K_{i,v}(x)\frac{(-\alpha_i)^v}{v!}=1 \quad\text{and}\quad \sum_{\mu=0}^{\infty}K_{j,\mu}(y)\frac{(-\alpha_j)^\mu}{\mu!}=1,$$

(iii) For any $x, y \in [0,\infty)$,

$$K_{i,v}(x) = -ixK_{i+m,v-1}(x) \quad \text{and} \quad K_{j,\mu}(y) = -jyK_{j+n,\mu-1}(y),$$

where $i+m$, $j+n$ are natural numbers and m, n are constants independent of v, μ.

Now let $a = (a_{ij}) = \begin{pmatrix} -1 & 0 & 0 & . \\ 0 & -1 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix}$ and let $x = (x_{ij})$ be given by (1.3) in Example 1.4. Observe now that $\sum_{(i,j)\in\mathbb{N}^2} |a_{ij}| = 2$ and $\sum_{(i,j)\in\mathbb{N}^2} a_{nk} = -2$. Using the operators $L_{ij}(f;x,y)$, we introduce the following positive linear operators:

$$B_{ij}(f;x,y) = x_{ij}L_{ij}(f;x,y).$$

Also, take $\rho_1(x,y) = 1 + x^2 + y^2$ and $\rho_2(x,y)$ arbitrary such that the condition $\lim_{\sqrt{x^2+y^2}\to\infty} \frac{\rho_1(x,y)}{\rho_2(x,y)} = 0$ is satisfied. Then we obtain the test functions $F_0(x,y) = 1$, $F_1(x,y) = x$, $F_2(x,y) = y$ and $F_3(x,y) = x^2 + y^2$. Now we claim that

$$st_2\text{-}\lim_{i,j} \left\| \sum_{m=1}^{i}\sum_{n=1}^{j} |a_{i-m+1j-n+1}|\, B_{mn}(F_r) - F_r \right\|_{\rho_1} = 0 \quad \text{for each } r = 0,1,2,3. \tag{4.1}$$

First observe that

$$\begin{aligned} B_{ij}(F_0;x,y) &= x_{ij}F_0(x,y), \\ B_{ij}(F_1;x,y) &= x_{ij}\frac{\alpha_i}{\beta_i}iF_1(x,y), \\ B_{ij}(F_2;x,y) &= x_{ij}\frac{\alpha_j}{\gamma_j}jF_2(x,y), \\ B_{ij}(F_3;x,y) &= x_{ij}\left[\frac{\alpha_i^2}{\beta_i^2}i(i+m)x^2 + \frac{\alpha_i}{\beta_i^2}ix + \frac{\alpha_j^2}{\gamma_j^2}j(j+n)y^2 + \frac{\alpha_j}{\gamma_j^2}jy\right]. \end{aligned}$$

So,

$$\left| \sum_{m=1}^{i}\sum_{n=1}^{j} |a_{i-m+1j-n+1}|\, B_{mn}(F_0;x,y) - F_0(x,y) \right| = \left| \sum_{m=1}^{i}\sum_{n=1}^{j} |a_{i-m+1j-n+1}|\, x_{ij} - 1 \right|$$

and then

$$\left(\left| \sum_{m=1}^{i}\sum_{n=1}^{j} |a_{i-m+1j-n+1}|\, x_{ij} - 1 \right| \right) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & . \\ 0 & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix}.$$

Since $\sup_{(x,y)\in[0,\infty)\times[0,\infty)} \frac{1}{1+x^2+y^2} < \infty$, we obtain

$$\begin{aligned} st_2\text{-}\lim_{i,j} & \left\| \sum_{m=1}^{i}\sum_{n=1}^{j} \left|a_{i-m+1j-n+1}\right| B_{mn}(F_0) - F_0 \right\|_{\rho_1} \\ &= st_2\text{-}\lim_{i,j} \sup_{(x,y)\in[0,\infty)\times[0,\infty)} \frac{\left|\sum_{m=1}^{i}\sum_{n=1}^{j} \left|a_{i-m+1j-n+1}\right| x_{ij} - 1\right|}{1+x^2+y^2} = 0, \end{aligned}$$

which guarantees that (4.1) holds true for $r = 0$. It is obvious that

$$\begin{aligned} & \left|\sum_{m=1}^{i}\sum_{n=1}^{j} \left|a_{i-m+1j-n+1}\right| B_{mn}(F_1;x,y) - F_1(x,y)\right| \\ &\qquad = \left|\sum_{m=1}^{i}\sum_{n=1}^{j} \left|a_{i-m+1j-n+1}\right| x_{ij}\frac{\alpha_i}{\beta_i} ix - x\right| \\ &\qquad = |x|\left|\sum_{m=1}^{i}\sum_{n=1}^{j} \left|a_{i-m+1j-n+1}\right| x_{ij}\frac{\alpha_i}{\beta_i} i - 1\right|; \end{aligned}$$

then, by virtue of (c), we can easily see that $st_2\text{-}\lim_{i,j} y_{ij} = 1$, where

$$(y_{ij}) = \left(\sum_{m=1}^{i}\sum_{n=1}^{j} \left|a_{i-m+1j-n+1}\right| x_{ij}\frac{\alpha_i}{\beta_i} i\right).$$

Also, since $\sup\limits_{(x,y)\in[0,\infty)\times[0,\infty)} \frac{|x|}{1+x^2+y^2} < \infty$, we get

$$\begin{aligned} st_2\text{-}\lim_{i,j} & \left\| \sum_{m=1}^{i}\sum_{n=1}^{j} \left|a_{i-m+1j-n+1}\right| B_{mn}(F_1) - F_1 \right\|_{\rho_1} \\ &= st_2\text{-}\lim_{i,j} \sup_{(x,y)\in[0,\infty)\times[0,\infty)} \frac{|x|\left|\sum\limits_{m=1}^{i}\sum\limits_{n=1}^{j} \left|a_{i-m+1j-n+1}\right| x_{ij}\frac{\alpha_i}{\beta_i} i - 1\right|}{1+x^2+y^2} = 0, \end{aligned}$$

which guarantees that (4.1) holds true for $r = 1$. Similarly we have

$$st_2\text{-}\lim_{i,j} \left\| \sum_{m=1}^{i}\sum_{n=1}^{j} \left|a_{i-m+1j-n+1}\right| B_{mn}(F_2) - F_2 \right\|_{\rho_1} = 0.$$

Finally, since

$$\begin{aligned}
&\left|\sum_{m=1}^{i}\sum_{n=1}^{j}\left|a_{i-m+1j-n+1}\right| B_{mn}(F_3;x,y)-F_3(x,y)\right| \\
&= \Bigg| \sum_{m=1}^{i}\sum_{n=1}^{j}\left|a_{i-m+1j-n+1}\right| x_{ij}\left[\frac{\alpha_i^2}{\beta_i^2}i(i+m)x^2+\frac{\alpha_i}{\beta_i^2}ix+\frac{\alpha_j^2}{\gamma_j^2}j(j+n)y^2\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left.+\frac{\alpha_j}{\gamma_j^2}jy\right]-x^2-y^2\Bigg| \\
&= \left|x^2\right|\left|\sum_{m=1}^{i}\sum_{n=1}^{j}\left|a_{i-m+1j-n+1}\right| x_{ij}\frac{\alpha_i^2}{\beta_i^2}i(i+m)-1\right| \\
&\quad+\left|y^2\right|\left|\sum_{m=1}^{i}\sum_{n=1}^{j}\left|a_{i-m+1j-n+1}\right| x_{ij}\frac{\alpha_j^2}{\gamma_j^2}j(j+n)-1\right| \\
&\quad+|x|\left|\sum_{m=1}^{i}\sum_{n=1}^{j}\left|a_{i-m+1j-n+1}\right| x_{ij}\frac{\alpha_i}{\beta_i^2}i\right|+|y|\left|\sum_{m=1}^{i}\sum_{n=1}^{j}\left|a_{i-m+1j-n+1}\right| x_{ij}\frac{\alpha_j}{\gamma_j^2}j\right|,
\end{aligned}$$

and because of (c), we can easily see that

$$\begin{aligned}
&st_2\text{-}\lim_{i,j}\sum_{m=1}^{i}\sum_{n=1}^{j}\left|a_{i-m+1j-n+1}\right| x_{ij}\frac{\alpha_i^2}{\beta_i^2}i(i+m)=1, \\
&st_2\text{-}\lim_{i,j}\sum_{m=1}^{i}\sum_{n=1}^{j}\left|a_{i-m+1j-n+1}\right| x_{ij}\frac{\alpha_j^2}{\gamma_j^2}j(j+n)=1,
\end{aligned} \tag{4.2}$$

and since $\lim\limits_{i\to\infty}\frac{1}{b_i}=0$ and $\lim\limits_{j\to\infty}\frac{1}{\gamma_j}=0$ from (a) and (c), we get

$$\begin{aligned}
&st_2\text{-}\lim_{i,j}\sum_{m=1}^{i}\sum_{n=1}^{j}\left|a_{i-m+1j-n+1}\right| x_{ij}\frac{\alpha_i}{\beta_i^2}i=0, \\
&st_2\text{-}\lim_{i,j}\sum_{m=1}^{i}\sum_{n=1}^{j}\left|a_{i-m+1j-n+1}\right| x_{ij}\frac{\alpha_j}{\gamma_j^2}j=0.
\end{aligned} \tag{4.3}$$

Using (4.2) and (4.3) and since

$$\begin{aligned}
&\sup_{(x,y)\in[0,\infty)\times[0,\infty)}\frac{|x|^2}{1+x^2+y^2}<\infty, &&\sup_{(x,y)\in[0,\infty)\times[0,\infty)}\frac{|y|^2}{1+x^2+y^2}<\infty, \\
&\sup_{(x,y)\in[0,\infty)\times[0,\infty)}\frac{|x|}{1+x^2+y^2}<\infty, &&\sup_{(x,y)\in[0,\infty)\times[0,\infty)}\frac{|y|}{1+x^2+y^2}<\infty,
\end{aligned}$$

we can write

$$st_2\text{-}\lim_{i,j}\left\|\sum_{m=1}^{i}\sum_{n=1}^{j}\left|a_{i-m+1j-n+1}\right|B_{mn}(F_3)-F_3\right\|_{\rho_1}$$

$$\leq st_2\text{-}\lim_{i,j}\sup_{(x,y)\in[0,\infty)\times[0,\infty)}\frac{\left|x^2\right|\left|\sum\limits_{m=1}^{i}\sum\limits_{n=1}^{j}\left|a_{i-m+1j-n+1}\right|x_{ij}\frac{\alpha_i^2}{\beta_i^2}i(i+m)-1\right|}{1+x^2+y^2}$$

$$+\,st_2\text{-}\lim_{i,j}\sup_{(x,y)\in[0,\infty)\times[0,\infty)}\frac{\left|y^2\right|\left|\sum\limits_{m=1}^{i}\sum\limits_{n=1}^{j}\left|a_{i-m+1j-n+1}\right|x_{ij}\frac{\alpha_j^2}{\gamma_j^2}j(j+n)-1\right|}{1+x^2+y^2}$$

$$+\,st_2\text{-}\lim_{i,j}\sup_{(x,y)\in[0,\infty)\times[0,\infty)}\frac{|x|\left|\sum\limits_{m=1}^{i}\sum\limits_{n=1}^{j}\left|a_{i-m+1j-n+1}\right|x_{ij}\frac{\alpha_i}{\beta_i^2}i\right|}{1+x^2+y^2}$$

$$+\,st_2\text{-}\lim_{i,j}\sup_{(x,y)\in[0,\infty)\times[0,\infty)}\frac{|y|\left|\sum\limits_{m=1}^{i}\sum\limits_{n=1}^{j}\left|a_{i-m+1j-n+1}\right|x_{ij}\frac{\alpha_j}{\gamma_j^2}j\right|}{1+x^2+y^2}$$

So, our claim (4.1) holds true for each $i=0,1,2,3$. The sequence (B_{ij}) satisfies all hypothesis of Theorem 2.5 and we immediately see that

$$st_2\text{-}\lim_{i,j}\left\|\sum_{m=1}^{i}\sum_{n=1}^{j}\left|a_{i-m+1j-n+1}\right|B_{mn}(f)-f\right\|_{\rho_2}=0,\quad\text{for all } f\in C_{\rho_1}.$$

However, since $\left\|B_{ij}(F_0)-F_0\right\|_{\rho_1}=|x_{ij}-1|$, a sequence $\left(\left\|B_{ij}(F_0)-F_0\right\|_{\rho_1}\right)$ does not converge in Pringsheim's and statistical senses. So, Theorem 2.1 and Theorem 2.2 do not work for the sequence (B_{ij}). Also, since

$$P\text{-}\lim_{i,j}\left\|\frac{1}{ij}\sum_{m=k}^{k+i-1}\sum_{n=l}^{l+j-1}B_{mn}(F_0)-F_0\right\|_{\rho_1}\neq 0,\quad n\in\mathbb{N},$$

Theorem 2.3 does not work for the sequence (B_{ij}), either.

5. Conclusion

We introduced a new method of summability, namely, statistical K_a^2-convergence and obtained a Korovkin type approximation theorem for double sequences on two-dimensional weighted spaces via this method. We obtained the rate of statistical K_a^2-convergence and finally we presented an application that shows that our result is stronger than studied before. We note that if conditions (1.1), (1.2) and $\sum_{(i,j)\in\mathbb{N}^2}a_{ij}=1$ hold, then the four-dimensional matrix A given via a non-negative double sequence $a=(a_{ij})$ is RH-regular. In that case the statistical K_a^2-convergence is a special case of statistical A-summability for double sequences ([4, 13]). But in this paper we consider any double sequence $a=(a_{ij})$ and conditions (1.1) and (1.2) are satisfied. Hence, our results here are meaningful.

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