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A CHARACTERIZATION OF STONE AND LINEAR HEYTING ALGEBRAS

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Abstract. An important problem in the variety of Heyting algebras $\mathcal{H}$ is to find new characterizations which allow us to determinate if a given $H \in \mathcal{H}$ is linear or Stone. In this work we present two Heyting algebras, H^{ns} and H^{snl}, such that: (a) a Heyting algebra H is a Stone–Heyting algebra if and only if H^{ns} cannot be embedded in H, and (b) H is a linear Heyting algebra if and only if neither H^{ns} nor H^{snl} can be embedded in H.

1. Introduction

It is well known that Heyting algebras are the algebraic models of intuitionistic logic. On the other hand there are different subvarieties of Heyting algebras which are the logical counterparts of certain intermediate logics. Two important examples are the linear Heyting algebras and Stone–Heyting algebras. The first one corresponds to the logic of Gödel and Dummett ([3], [4]) and were studied by several authors (see, for instance, [6]). Stone algebras are pseudo-complemented distributive lattices satisfying the so-called weak excluded middle, $\neg a \vee \neg\neg a = 1$. These structures were introduced by Grätzer and Schmidt in [5]. An important and useful tool in universal algebra is to determine, by means of certain embeddings, whether an algebra belongs to a given subvariety. A classical example is the variety of lattices and the subvarieties of modular and distributive lattices. Indeed, in this case the result establishes that a lattice L is modular if and only if N_5 cannot be embedded in L, and L is distributive if and only if neither N_5 nor M_5 can be embedded in L [2]. In this paper we show such embedding results for the varieties of linear Heyting algebras and Stone–Heyting algebras. The main results are established in Theorems 3.3 and 4.4 by showing that a Heyting algebra H is a Stone algebra if an only if the Heyting algebra H^{ns}, determined by the Boolean algebra with four elements with a new one added, cannot be embedded in H, while a Heyting algebra H is linear if and only if neither H^{ns} nor H^{snl} can be embedded into H, where H^{snl} is the Boolean algebra with four elements with a new one and zero added. In order to obtain this last result we need to introduce a new class of Heyting algebras called *DL algebras.* Finally we describe these results by means of

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the partially ordered set associated to the lattice of subvarieties of the variety of Heyting algebras.

2. Preliminaries

The aim of this section is to establish some basic facts about Heyting algebras which are needed in the paper. Recall that a *Heyting algebra* is a bounded distributive lattice $(H, \vee, \wedge, 0, 1)$ such that for every $x, y \in H$ the set $\{z \in H : z \wedge x \leq y\}$ has a last element, denoted by $x \to y$. In particular, the *Heyting negation* of an element $x \in H$, called also intuitionistic, is defined by $\neg x = x \to 0$, which is the greatest element $z \in H$ such that $x \wedge z = 0$. Hence every Heyting algebra is a *pseudocomplemented distributive lattice* endowed with this negation. The class of Heyting algebras determines a variety in the sense of universal algebra and hence is an equational class in the language $(\vee, \wedge, \to, 0, 1)$, given by the following equations (see, for instance, [1]):

(H0) H is a bounded distributive lattice.
(H1) $x \wedge (x \to y) = x \wedge y$.
(H2) $x \wedge (y \to z) = x \wedge ((x \wedge y) \to (x \wedge z))$.
(H3) $(x \wedge y) \to x = 1$.

We shall denote by $\mathcal{H}$ the variety of Heyting algebras.

Let H be a Heyting algebra and let $z \in H$. Recall that z is called a *dense element* provided that $\neg z = 0$. It is immediate to see that z is dense if and only if there is $x \in H$ such that $z = x \vee \neg x$. We shall denote by $D(H)$ the set of dense elements of H. Since $\neg(x \wedge y) = x \to \neg y$, it follows that $D(H)$ is a filter of H.

Lemma 2.1. *Let H be a Heyting algebra. Then the following conditions are satisfied:*

(i) *If $x \in D(H)$ and $y \in H$ then $x \to \neg y = \neg y$.*
(ii) *If $x, y \in H$ then $(x \to y) \to (y \to x) = (y \to x)$.*

Proof. To prove (i), let x be a dense element of H. It is plain that $x \wedge \neg y \leq \neg y$. Take $z \in H$ such that $x \wedge z \leq \neg y$. Then $x \wedge z \wedge y = 0$. Since $x \in D(H)$ it follows that $z \wedge y = 0$ and hence $z \leq \neg y$. To prove (ii), let x, y be elements in H. We first prove that $(x \to y) \wedge (y \to x) \leq (y \to x)$. Indeed, this inequality is equivalent to proving that $(x \to y) \wedge (y \to x) \wedge y \leq x$. Since $(x \to y) \wedge (y \to x) \wedge y = y \wedge (y \to x) = y \wedge x \leq x$ the results follows. Next we take $z \in H$ such that $(x \to y) \wedge z \leq (y \to x)$. It follows that $(x \to y) \wedge z \wedge y = z \wedge y \leq x$ and hence $z \leq (y \to x)$. Therefore $(x \to y) \to (y \to x) = (y \to x)$. □

Recall that an algebra A is *subdirectly irreducible* if for every subdirect embedding $f : A \to \prod_{i \in I} A_i$ there is $i \in I$ such that $\pi_i \circ f : A \to A_i$ is an isomorphism, where π_i denotes the canonical projection from $\prod_{i \in I} A_i$ onto A_i. The following characterization of subdirectly irreducible Heyting algebras is well known and will be useful through this paper.

Proposition 2.2 ([2, p. 66, Exercise 9]). *Let H be a Heyting algebra. Then H is a subdirectly irreducible algebra if and only if there is a Heyting algebra $\overline{H}$ such that $H = \overline{H}\dagger 1$, where $\dagger$ denotes the ordinal sum, or equivalently, $1 \in H$ is join-irreducible (if $a \vee b = 1$ then either $a = 1$ or $b = 1$).*

3. Stone–Heyting algebras

Recall that a a pseudo-complemented distributive lattice $(L, \vee, \wedge, \neg, 0, 1)$ is called a *Stone algebra* provided that $\neg a \vee \neg\neg a = 1$ for all $a \in L$. A *Stone–Heyting algebra* is a Heyting algebra H such that the underlying pseudo-complemented distributive lattice is a Stone algebra. In this case the pseudocomplement $\neg$ coincides with the intuitionistic negation defined in H.

The variety of Stone–Heyting algebras will be denoted by $\mathcal{H}^S$. It is worthwhile to point out that although the variety of Stone algebras is generated by the chains having either two or three elements, this is not true for the variety $\mathcal{H}^S$. Indeed we have the following result.

Theorem 3.1. *Let $V(\mathbf{3})$ be the subvariety of $\mathcal{H}$ generated by the Heyting chain with three elements. Then $V(\mathbf{3})$ is characterized by the equation*

$$(x \vee \neg x) \vee (y \vee \neg y) \vee [(x \to y) \wedge (y \to x)] = 1. \qquad (*)$$

Proof. It is clear that $\mathbf{3}$ satisfies the above equation. Therefore every algebra in $V(\mathbf{3})$ satisfies $(*)$. In order to complete the proof it is enough to see that if H is a subdirectly irreducible algebra satisfying $(*)$ then H has three elements. By Proposition 2.2 we know that $H = \overline{H}\dagger 1$. Let $\overline{1}$ be the top element of $\overline{H}$. It is enough to prove that $H = \{0, \overline{1}, 1\}$. Indeed, take $x \in H$. Suppose that $x \notin \{0, \overline{1}, 1\}$. Let $y = \overline{1}$. By taking into account that $x \neq y$ and 1 is a join irreducible element of H we infer from $(*)$ that either $x \vee \neg x = 1$ or $y \vee \neg y = 1$. Since the second equality is impossible we infer that either $x = 0$ or $x = 1$, a contradiction. □

Note that a chain having more than three elements does not satisfy equation $(*)$ in the above theorem. In particular, $\mathcal{H}^S$ properly contains $V(\mathbf{3})$, since every finite chain belongs to $\mathcal{H}^S$.

Remark 3.2. The algebras belonging to the subvariety $V(\mathbf{3})$ are called *three-valued Heyting algebras* and were studied by L. Monteiro in [7]. It is shown in that paper that this variety is characterized by the equation

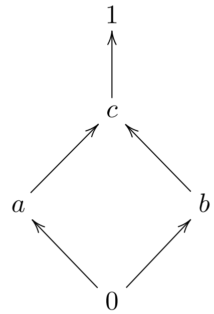
$$((x \to z) \to y) \to (((y \to x) \to y) \to y) = 1.$$

Note that this equation involves only the Heyting implication and depends on three variables x, y, z, while the equation given in the previous theorem depends on two variables x, y and all the operations associated to the language of Heyting algebras.

We denote by H^{ns} the Heyting algebra determined by the ordinal sum $\overline{B_4}\dagger 1$, where B_4 is the Boolean algebra with four elements. The Hasse diagram of this

Heyting algebra is given by the following picture:

H^{ns} :



Theorem 3.3. *Let H be a Heyting algebra. Then H is a Stone algebra if and only if there does not exist an embedding $J_1 : H^{ns} \to H$ of Heyting algebras.*

Proof. Assume first that H is a Stone algebra and assume on the contrary that $J_1 : H^{ns} \to H$ is an embedding of Heyting algebras. It follows that the image of J_1 is a subalgebra of H isomorphic to H^{ns}. Since H is a Stone algebra this would imply that H^{ns} is also a Stone algebra, a contradiction. For the converse, assume that H is not a Stone algebra. Then there is $h \in H$ such that $\neg h \vee \neg\neg h \neq 1$. We claim that the subalgebra generated by $\neg h$ is $\{0, \neg h, \neg\neg h, (\neg h \vee \neg\neg h), 1\}$. Indeed, since $(\neg h \vee \neg\neg h) \in D(H)$ it follows from Lemma 2.1 (i) that $(\neg h \vee \neg\neg h) \to \neg h = \neg h$ and $(\neg h \vee \neg\neg h) \to \neg\neg h = \neg\neg h$, while $(\neg h \vee \neg\neg h) \to 0 = 0$. On the other hand, it is immediate to see that the following identities hold in every Heyting algebra:

(a) $\neg h \to 0 = \neg\neg h$,
(b) $\neg\neg h \to 0 = \neg h$,
(c) $\neg h \to \neg\neg h = \neg\neg h$, and
(d) $\neg\neg h \to \neg h = \neg h$,

which proves the claim. Since this subalgebra is isomorphic to H^{ns}, we obtain a contradiction. ☐

Proposition 3.4. *Let H be a subdirectly irreducible Stone–Heyting algebra. Then $D(H) = H \backslash \{0\}$.*

Proof. Let H be a subdirectly irreducible Stone–Heyting algebra and let $x \in H$ such that $x \neq 0$. Since H is a Stone algebra, we have $\neg x \vee \neg\neg x = 1$. By taking into account that H is subdirectly irreducible, it follows from Proposition 2.2 that either $\neg x = 1$ or $\neg\neg x = 1$. If $\neg x = 1$, then $x = 0$, which is a contradiction. Thus $\neg\neg x = 1$, and consequently $\neg x = 0$. ☐

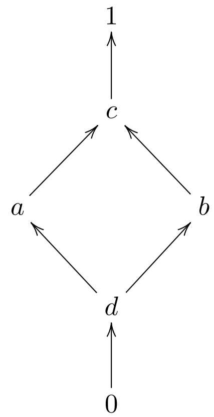
4. Linear Heyting algebras

Recall that a Heyting algebra H is said to be a *linear Heyting algebra* provided the following identity is satisfied:

$$(a \to b) \vee (b \to a) = 1 \tag{L}$$

for all $a, b \in H$. We shall denote by $\mathcal{H}^L$ the variety of linear Heyting algebras. Every linear Heyting algebra is a Stone algebra; in particular, $\mathcal{H}^S$ is a subvariety of $\mathcal{H}^L$. Indeed, let H be an algebra in $\mathcal{H}^L$. Replacing b by $\neg a$ in (L) we obtain $(a \to \neg a) \vee (\neg a \to a) = 1$. Since $a \to \neg a = \neg a$ and $\neg a \to a = \neg\neg a$, we arrive to the identity $\neg a \vee \neg\neg a = 1$, thus proving that H is a Stone algebra. Moreover, $\mathcal{H}^L$ is a proper subvariety of $\mathcal{H}^S$. Indeed, the Heyting algebra H^{snl}, given by the following Hasse diagram, belongs to $\mathcal{H}^S$ and does not belong to $\mathcal{H}^L$.

H^{snl} :



Indeed, since every element $h \in H^{snl} \backslash \{0\}$ is dense it follows that H^{snl} is a Stone algebra, while $(a \to b) \vee (b \to a) = b \vee a = c \neq 1$ implies that H^{snl} does not belong to $\mathcal{H}^L$.

Definition 4.1. Let H be a Heyting algebra. We say that H is a *DL algebra* provided the dense elements of H satisfy equation (L), i.e., H satisfies the equation $((x \vee \neg x) \to (y \vee \neg y)) \vee ((y \vee \neg y) \to (x \vee \neg x)) = 1$.

The variety of DL algebras will be denoted by $\mathcal{H}^D$.

It is easy to see that $\mathcal{H}^L$ is a proper subvariety of $\mathcal{H}^D$. Indeed, $H^{ns} \in \mathcal{H}^D$ and $H^{ns} \notin \mathcal{H}^L$.

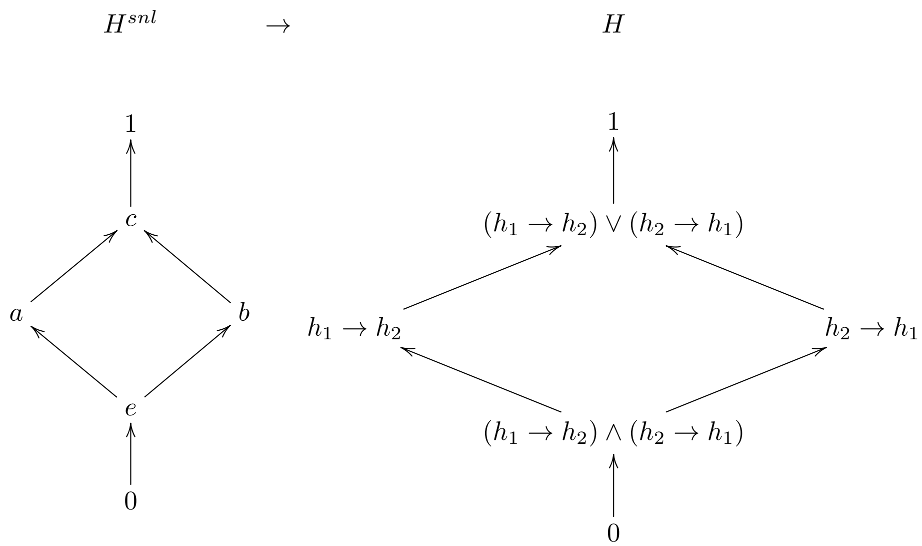
Theorem 4.2. *Let H be a Heyting algebra. Then $H \in \mathcal{H}^D$ if and only if there does not exist an embedding $J_2 : H^{snl} \to H$ of Heyting algebras.*

Proof. Assume first that H is a DL algebra. Since H^{ns} is not a DL algebra, it is plain that it cannot be isomorphic to any subalgebra of H and hence there is no embedding from H^{snl} into H. For the converse, assume on the contrary that $H \notin \mathcal{H}^D$. Then there are elements $h_1, h_2 \in D(H)$ such that $(h_1 \to h_2) \vee (h_2 \to h_1) \neq 1$. Note that h_1 and h_2 are incomparable, which implies that $h_1 \to h_2$ and $h_2 \to h_1$ are also incomparable. Since $D(H)$ is a filter of H, it follows that $h_1 \to h_2$, $h_2 \to h_1$, $((h_1 \to h_2) \vee (h_2 \to h_1))$ and $((h_1 \to h_2) \wedge (h_2 \to h_1))$ are also in $D(H)$.

We define the following map:

$$\begin{aligned} j_2 : H^{snl} &\to H \\ a &\to (h_1 \to h_2) \\ b &\to (h_2 \to h_1) \\ c &\to ((h_1 \to h_2) \vee (h_2 \to h_1)) \\ 1 &\to 1 \\ e &\to (h_1 \to h_2) \wedge (h_2 \to h_1) \\ 0 &\to 0. \end{aligned}$$

j_2 acts according to the following picture:



because $h_1 \to h_2$ and $h_2 \to h_1$ are incomparable. It is clear that j_2 is a lattice embedding. Let $x = h_1 \to h_2$ and $y = h_2 \to h_1$. By Lemma 2.1 (ii) it follows that $x \to y = y$ and $y \to x = x$. Therefore $(x \vee y) \to x = x$, $(x \vee y) \to y = y$, $x \to (x \wedge y) = y$ and $y \to (x \wedge y) = x$, which implies that j_2 is an embedding of Heyting algebras and we arrive to a contradiction. □

Theorem 4.3. $\mathcal{H}^L = \mathcal{H}^D \cap \mathcal{H}^S$.

Proof. We know that $\mathcal{H}^L \subseteq \mathcal{H}^D \cap \mathcal{H}^S$. To prove the reverse inclusion it is enough to show that every subdirectly irreducible algebra H in $\mathcal{H}^D \cap \mathcal{H}^S$ belongs to $\mathcal{H}^L$. Let x, y be in H. If $x = 0$ or $y = 0$, then it is clear that $(x \to y) \vee (y \to x) = 1$. Assume that $x \neq 0$ and $y \neq 0$. Then according to Proposition 3.4, x and y are both dense elements. Thus $x = x \vee \neg x$ and $y = y \vee \neg y$. Since $H \in \mathcal{H}^D$, we conclude that $(x \to y) \vee (y \to x) = ((x \vee \neg x) \to (y \vee \neg y)) \vee ((y \vee \neg y) \to (x \vee \neg x)) = 1$. □

Theorem 4.4. *Let $H \in \mathcal{H}$. Then $H \in \mathcal{H}^{\mathcal{L}}$ if and only if neither H^{ns} nor H^{snl} can be embedded in H.*

Proof. Assume first that H is a linear Heyting algebra. By taking into account that both H^{ns} and H^{snl} are not in $\mathcal{H}^{\mathcal{L}}$, it is plain that neither H^{ns} nor H^{snl} can be isomorphic to any subalgebra of H. To prove the converse, suppose on the contrary that H does not belong to $\mathcal{H}^{\mathcal{L}}$. Hence, by Theorem 4.3, $H \notin \mathcal{H}^D$ or $H \notin \mathcal{H}^S$. If $H \notin \mathcal{H}^S$ it follows from Theorem 3.3 that there exists an embedding $J_1 : H^{ns} \to H$ of Heyting algebras, in contradiction with our hypothesis. Analogously, if $H \notin \mathcal{H}^D$ we can find, according to Theorem 4.2, an embedding $J_2 : H^{snl} \to H$ of Heyting algebras, a contradiction. □

Theorems 3.3, 4.2 and 4.4 allow us to obtain another result which shows the connection between these embedding results and the the partial order given in the lattice $L(\mathcal{H})$ of subvarieties of Heyting algebras. Given a class A of Heyting algebras we shall denote by $\mathcal{V}(A)$ the subvariety of $\mathcal{H}$ generated by A.

Proposition 4.5.

(1) *The subvariety $\mathcal{V}(\mathcal{H}^S \cup \{H^{ns}\})$ is the unique successor of $\mathcal{H}^S$ in $L(\mathcal{H})$.*
(2) *The subvariety $\mathcal{V}(\mathcal{H}^D \cup \{H^{snl}\})$ is the unique successor of $\mathcal{H}^D$ in $L(\mathcal{H})$.*
(3) *The subvariety $\mathcal{H}^{\mathcal{L}}$ has exactly two successors, which are $\mathcal{V}(\mathcal{H}^L \cup \{H^{snl}\})$ and $\mathcal{V}(\mathcal{H}^L \cup \{H^{ns}\})$.*

Proof. (1) Let $\mathcal{W}$ be a subvariety of $\mathcal{H}$ that strictly contains $\mathcal{H}^S$ and let $H \in (\mathcal{W} - \mathcal{H}^S)$. Since H is not a Stone–Heyting algebra, it follows from Theorem 3.3 that there exists an embedding $J_1 : H^{ns} \to H$ of Heyting algebras. Since H^{ns} is isomorphic to a subalgebra of H, we have that $H^{ns} \in \mathcal{W}$. Consequently $\mathcal{V}(\mathcal{H}^S \cup \{H^{ns}\}) \subseteq \mathcal{W}$.

(2) Let $\mathcal{W}$ be a subvariety of $\mathcal{H}$ that strictly contains $\mathcal{H}^D$ and let $H \in (\mathcal{W} - \mathcal{H}^D)$. Since H is not a DL algebra, it follows from Theorem 4.2 that there exists an embedding $J_2 : H^{snl} \to H$ of Heyting algebras. Since H^{snl} is isomorphic to a subalgebra of H, we have that $H^{snl} \in \mathcal{W}$, thus proving the inclusion $\mathcal{V}(\mathcal{H}^D \cup \{H^{snl}\}) \subseteq \mathcal{W}$.

(3) We first claim that $H^{ns} \notin \mathcal{V}(\mathcal{H}^L \cup \{H^{snl}\})$ and $H^{snl} \notin \mathcal{V}(\mathcal{H}^L \cup \{H^{ns}\})$. Indeed, otherwise we would have in the first case that H^{ns} should be a Stone algebra, while in the second case H^{snl} should be a DL algebra, both of which yield a contradiction. Therefore $\mathcal{V}(\mathcal{H}^L \cup \{H^{snl}\})$ and $\mathcal{V}(\mathcal{H}^L \cup \{H^{ns}\})$ are incomparable subvarieties of $\mathcal{H}$ which strictly contain $\mathcal{H}^{\mathcal{L}}$. Let $\mathcal{W}$ be a subvariety of $\mathcal{H}$ that strictly contains $\mathcal{H}^L$ and let $H \in (\mathcal{W} - \mathcal{H}^L)$. Since H is not a linear Heyting algebra, it follows from Theorem 4.4 that either H^{ns} is isomorphic to a subalgebra of H or H^{snl} is isomorphic to a subalgebra of H. In the first case we have $H^{ns} \in \mathcal{W}$ and consequently $\mathcal{V}(\mathcal{H}^L \cup \{H^{ns}\}) \subseteq \mathcal{W}$; in the second case, $H^{snl} \in \mathcal{W}$, which proves that $\mathcal{V}(\mathcal{H}^L \cup \{H^{snl}\}) \subseteq \mathcal{W}$. □

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