

CONFORMAL VECTOR FIELDS ON STATISTICAL MANIFOLDS

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ABSTRACT. Introducing the conformal vector fields on a statistical manifold, we present necessary and sufficient conditions for a vector field on a statistical manifold to be conformal. After presenting some examples, we classify the conformal vector fields on two famous statistical manifolds. Considering three statistical structures on the tangent bundle of a statistical manifold, we study the conditions under which the complete and horizontal lifts of a vector field can be conformal on these structures.

1. INTRODUCTION

Nowadays, information geometry as a combination of statistics and differential geometry has an effective role in science. Some of its vast applications can be found in image processing, physics, computer science and machine learning [4, 9, 13, 12, 31]. It is a realm that makes it possible to illustrate statistical objects as geometric ones by the way of capturing their geometric properties. Rigid objects in the sense of coordinate transformation are a favourite among differential geometers. So, observing the statistical spaces from the doorway of differential geometry makes it convenient to study the statistical behaviors profoundly. A fundamental detailed survey on information geometry can be found in the monograph [5].

For an open subset Θ of \mathbb{R}^n and a sample space Ω with parameter $\theta = (\theta^1, \dots, \theta^n)$, we call the set of probability density functions

$$S = \{p(x; \theta) : \int_{\Omega} p(x; \theta) = 1, p(x; \theta) > 0, \theta \in \Theta \subseteq \mathbb{R}^n\}$$

a *statistical model*. For a statistical model S , the semi-definite *Fisher information matrix* $g(\theta) = [g_{ij}(\theta)]$ is defined as

$$g_{ij}(\theta) := \int_{\Omega} \partial_i \ell_{\theta} \partial_j \ell_{\theta} p(x; \theta) dx = E_p[\partial_i \ell_{\theta} \partial_j \ell_{\theta}], \quad (1.1)$$

where $\ell_{\theta} = \ell(x; \theta) := \log p(x; \theta)$, $\partial_i := \frac{\partial}{\partial \theta^i}$, and $E_p[f]$ is the expectation of $f(x)$ with respect to $p(x; \theta)$. The space S equipped with such information matrices is called a *statistical manifold* in the literature.

2020 *Mathematics Subject Classification.* 53B12, 53C18.

Key words and phrases. Conformal vector field, generalized distribution manifold, statistical connection, tangent bundle.

Historically, Fisher was the first to introduce the relation (1.1) in 1920 [16]. It is easy to see that if g is positive-definite and all of its components are converging to real numbers, then (S, g) will be a Riemannian manifold, and g is called a *Fisher metric* on S with components

$$g_{ij}(\theta) = \int_{\Omega} \partial_i p(x; \theta) \partial_j \ell_{\theta} dx = \int_{\Omega} \frac{1}{p(x; \theta)} \partial_i p(x; \theta) \partial_j p(x; \theta) dx.$$

Rao was the first to study the above metric in 1945 [28]. In 1982, Amari [2] studied a parametric family of torsion-free connections $\nabla^{(\alpha)}$ with respect to $p(x; \theta)$ (called α -connections) by using the Christoffel symbols

$$\Gamma_{ijk}^{(\alpha)} = g(\nabla_{\partial_i}^{\alpha} \partial_j, \partial_k) := E_p \left[\left(\partial_i \partial_j \ell_{\theta} + \frac{1 - \alpha}{2} \partial_i \ell_{\theta} \partial_j \ell_{\theta} \right) (\partial_k \ell_{\theta}) \right],$$

where $\alpha \in \mathbb{R}$. Indeed, by introducing α -connections, Amari provided a differential geometrical framework for analyzing statistical problems related to multiparameter families of distributions and introduced α -geometry on statistical manifolds. α -geometry measures second-order information loss and the second-order efficiency of an estimator. The α -geometry of the Gaussian, Gamma, McKay bivariate gamma, Weibull and Freund bivariate exponential manifolds are studied by Amari [3], Arwini and Dodson [6] and Cao, Sun and Wang [11]. An interesting feature of the Gaussian and the Weibull manifolds is that they have negative constant Gaussian curvature [6, 11]. Also, one interesting fact is that several of the submanifolds of the Freund bivariate exponential manifold are α -flat [6]. The statistical manifolds whose α -curvature is negative constant have the same statistical properties as Gaussian and Weibull manifolds. In particular, some parameter's MLE in α -flat statistical manifolds has no second order information loss (see [2, 15] for more details). The question now arises statistically and geometrically. Indeed, that question is whether there may be other manifolds with constant Gaussian curvature or α -Gaussian curvature. In answer to this question, Yuan [34] proved that the generalized Gaussian statistical manifold has constant α -Gaussian curvature. Also, he introduced the p -dimensional statistical manifold (for any positive integer p) that is α -flat.

A *statistical manifold* is a triple (M, g, ∇) , where the manifold M is equipped with a statistical structure (g, ∇) containing a Riemannian metric g and an affine connection ∇ on M such that the covariant derivative ∇g is symmetric. The trivial example of statistical manifolds arises when one puts $\nabla := \nabla^{(0)}$, where $\nabla^{(0)}$ is the Levi-Civita connection of g . It can be checked that $\nabla^{(\alpha)}$ is torsion-free and that $\nabla^{(\alpha)} g$ is totally symmetric, so $(M, g, \nabla^{(\alpha)})$ is a statistical manifold. Lauritzen was the first to study such structures in 1987 [22].

Conformality is an interesting concept in several branches of mathematics, such as classical geometry, real and complex analysis, (semi-) Riemannian geometry and Finsler geometry. Also, it is a valuable concept in physics, in particular in conformal field theory and general relativity (see for instance [19, 27]). In fluid mechanics, aerodynamics, thermomechanics, electrostatics, elasticity, and elsewhere,

the solutions to the Laplace equation on complicated planar domains have extensive applications. An effective approach to the construction of such solutions is based on the use of conformal mappings [24]. Conformal vector fields are vector fields with flows preserving a given conformal class of metrics. These vector fields as the generalizations of conformal functions between Euclidean spaces and conformal maps between (semi-) Riemannian manifolds, are important matters of fact in Riemannian geometry (see for instance [8, 30]). Indeed, a smooth vector field X on a Riemannian manifold (M, g) is said to be a *conformal vector field* if there exists a smooth function ρ on M such that $L_X g = 2\rho g$. The function ρ is called the *potential function* of the conformal vector field X . If f is a constant function, X is called a *homothetic vector field*. A special category of homothetic vector fields is the set of Killing vector fields, where $\rho = 0$ (in the presence of a Riemannian metric g , their flow preserves g). These kinds of preserving the metric help to categorize the spaces in the sense of diffeomorphism and symmetry. The study of conformal vector fields on Riemannian manifolds and their tangent bundles is of interest to many researchers (see for instance [1, 18, 25, 32]).

Recently, the study of geometric concepts of statistical manifolds has been considered by many researchers (see e.g. [7, 10, 17, 20, 21, 35, 36, 37]). For instance, Sasakian geometry and symplectic geometry on statistical manifolds are introduced by [17] and [37], respectively. Also, Hasegawa and Yamauchi [20, 21] introduced the concepts of λ -conformally flat and conformally-projectively flat on statistical manifolds. The importance of conformal vector fields in Riemannian geometry and the concepts introduced in [20, 21] led to the idea of introducing conformal vector fields on statistical manifolds. Since the concept of conformal vector field is independent of the choice of linear connection, its introduction on statistical manifolds is similar to that on Riemannian ones and will not be valuable. So, we need to introduce a new concept that uses the statistical connection structure and gives us the definition of conformal vector field in the Riemannian case (for this reason we call it the conformal vector field on statistical manifolds). Therefore, the aim of this paper is to study the conformal geometry on statistical manifolds.

The organization of the paper is as follows. In Section 2 we recall some concepts on statistical manifolds and lift geometry on the tangent bundle of a manifold. Section 3 is devoted to the study of Lie derivatives of tensor fields on statistical manifolds. In Section 4 we introduce the conformal vector fields on statistical manifolds and we present some examples of them. Then we focus on two famous statistical manifolds (the general Gaussian distribution manifold and the 2-dimensional statistical manifold) and we determine the conformal vector fields on these manifolds. In the last section we consider three statistical structures on the tangent bundle of a Riemannian manifold and we find necessary and sufficient conditions under which the horizontal and complete lifts of a vector field can be conformal on these structures. Then we implement these conditions on an example, and also on the tangent bundle of the generalized Gaussian distribution manifold and 2-dimensional statistical manifold.

2. PRELIMINARIES

Let M be a smooth manifold with a Riemannian metric g and let ∇ be a linear symmetric connection. The triple (M, g, ∇) is called a *statistical manifold* if ∇g is symmetric for all $X, Y, Z \in \chi(M)$. In other words, $\nabla g = C$, where C is a symmetric tensor of degree $(0,3)$, namely, ∇g satisfies the Codazzi equations

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z) = (\nabla_Z g)(Y, X) = C(X, Y, Z) \quad \forall X, Y, Z \in \chi(M). \tag{2.1}$$

In this case, ∇ is called a *statistical connection*. When $C = 0$, we have the unique Levi-Civita connection $\nabla^{(0)}$. Now we define the *skewness operator* K of degree $(1,2)$ on M as follows:

$$K_X Y = \nabla_X Y - \nabla_X^{(0)} Y \quad \forall X, Y \in \chi(M). \tag{2.2}$$

It is easy to see that K satisfies the following relations:

$$\begin{aligned} \text{(i)} \quad & K_X Y = K_Y X, \\ \text{(ii)} \quad & g(K_X Y, Z) = g(Y, K_X Z), \\ \text{(iii)} \quad & C(X, Y, Z) = -2g(K_X Y, Z). \end{aligned} \tag{2.3}$$

The dual connection of a linear connection ∇ is defined by

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \overset{*}{\nabla}_X Z) \quad \forall X, Y, Z \in \chi(M).$$

It is known that if (M, g, ∇) is a statistical manifold, then $(M, g, \overset{*}{\nabla})$ is a statistical manifold as well. Moreover, we have $\nabla^{(0)} = \frac{1}{2}(\nabla + \overset{*}{\nabla})$. Using ∇ and $\overset{*}{\nabla}$, we have the family of α -connections as follows [35]:

$$\nabla^{(\alpha)} = \frac{1 + \alpha}{2} \nabla + \frac{1 - \alpha}{2} \overset{*}{\nabla}.$$

Let M be an n -dimensional manifold, let TM be its tangent bundle and let $\pi : TM \rightarrow M$ be the projection map. The space TTM can be split into two subspaces at every point (p, v) as follows:

$$T_{(p,v)} TTM = H_{(p,v)} TTM \oplus V_{(p,v)} TTM, \tag{2.4}$$

where $V_{(p,v)} TTM = \ker \pi_*|_{(p,v)}$ and $H_{(p,v)} TTM$ is a supplement space of $V_{(p,v)} TTM$. If $(x^i, y^i), i = 1, \dots, n$, is a local coordinate of TM , then $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}_{i=1}^n$ is a natural basis of TTM at every point (p, v) with the dual $\{dx^i, dy^i\}_{i=1}^n$. According to the splitting (2.4), TTM has the basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}_{i=1}^n$ with the dual $\{dx^i, \delta y^i\}_{i=1}^n$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}, \quad \delta y^i = dy^i + y^j \Gamma_{kj}^i dx^k,$$

and Γ_{ij}^k are the Christoffel symbols of a linear connection ∇ . From now on to simplify we use ∂_i, ∂_i and δ_i instead of $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}$ and $\frac{\delta}{\delta x^i}$, respectively. The Lie

brackets of $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}_{i=1}^n$ are given by

$$[\delta_i, \delta_j] = -y^r R_{ijr}^k \partial_{\bar{k}}, \quad [\delta_i, \partial_{\bar{j}}] = \Gamma_{ij}^k \partial_{\bar{k}}, \quad [\partial_{\bar{i}}, \partial_{\bar{j}}] = 0, \tag{2.5}$$

where R_{ijr}^k are the components of the curvature tensor of M given by

$$R_{ijr}^k = \partial_i \Gamma_{jr}^k - \partial_j \Gamma_{ir}^k + \Gamma_{jr}^h \Gamma_{hi}^k - \Gamma_{ir}^h \Gamma_{hj}^k. \tag{2.6}$$

It is known that the components of the curvature tensor of a statistical connection satisfy the following relations:

$$R_{ijr}^k = -R_{jir}^k, \quad R_{ijr}^k + R_{jri}^k + R_{rij}^k = 0.$$

Let $X = X^i \partial_i$ be a vector field on M . The vertical, horizontal and complete lifts of X are defined by

$$X^v = X^i \partial_{\bar{i}}, \quad X^h = X^i \delta_i, \quad X^c = X^i \partial_i + y^a (\partial_a X^i) \partial_{\bar{i}}.$$

The above lift operations are extended to the tensor algebra $\mathcal{J}(M)$ by the following rules [33]:

$$\begin{aligned} (P \otimes Q)^v &= P^v \otimes Q^v, \\ (P \otimes Q)^c &= P^c \otimes Q^v + P^v \otimes Q^c, \\ (P \otimes Q)^h &= P^v \otimes Q^h + P^h \otimes Q^v. \end{aligned} \tag{2.7}$$

In particular, for any tensor fields $P, Q \in \mathcal{J}_s^r(M)$ with $r = 0, 1$, we have

$$P^c(X_1^c, \dots, X_s^c) = (P(X_1, \dots, X_s))^c, \quad P^c(X_1^v, \dots, X_s^v) = 0, \tag{2.8}$$

and

$$\begin{aligned} Q^h(X_1^h, \dots, X_s^h) &= (Q(X_1, \dots, X_s))^h, \quad Q^h(X_1^v, \dots, X_s^v) = 0, \\ Q^h(X_1^h, \dots, X_{j-1}^h, X_j^v, X_{j+1}^h, \dots, X_s^h) &= (Q(X_1, \dots, X_s))^v. \end{aligned} \tag{2.9}$$

3. LIE DERIVATION AND STATISTICAL CONNECTION

First, we recall some concepts and notations on the Lie derivative and the covariant derivative of tensors.

Let $T \in \mathcal{J}_s^r(M)$ and let ∇ be a linear connection on M . Then, for each $X \in \mathcal{J}_0^1(M) = \chi(M)$, the *covariant derivative* of T along X is defined by (see [33])

$$\begin{aligned} (\nabla_X T)(\theta^1, \dots, \theta^r, X_1, \dots, X_s) &= X(T(\theta^1, \dots, \theta^r, X_1, \dots, X_s)) \\ &\quad - T(\nabla_X \theta^1, \dots, \theta^r, X_1, \dots, X_s) - \dots - T(\theta^1, \dots, \nabla_X \theta^r, X_1, \dots, X_s) \\ &\quad - T(\theta^1, \dots, \theta^r, \nabla_X X_1, \dots, X_s) - \dots - T(\theta^1, \dots, \theta^r, X_1, \dots, \nabla_X X_s), \end{aligned}$$

where $\theta^i \in \mathcal{J}_1^0(M)$, $X_j \in \mathcal{J}_0^1(M)$, $i = 1, \dots, r$ and $j = 1, \dots, s$. Also, the *Lie derivative* of T along X is defined by

$$\begin{aligned} (L_X T)(\theta^1, \dots, \theta^r, X_1, \dots, X_s) &= X(T(\theta^1, \dots, \theta^r, X_1, \dots, X_s)) \\ &\quad - T(L_X \theta^1, \dots, \theta^r, X_1, \dots, X_s) - \dots - T(\theta^1, \dots, L_X \theta^r, X_1, \dots, X_s) \\ &\quad - T(\theta^1, \dots, \theta^r, L_X X_1, \dots, X_s) - \dots - T(\theta^1, \dots, \theta^r, X_1, \dots, L_X X_s). \end{aligned}$$

In the local expression we have

$$\begin{aligned} \nabla_m T_{j_1 \dots j_s}^{i_1 \dots i_r} &:= (\nabla_{\partial_m} T)_{j_1 \dots j_s}^{i_1 \dots i_r} := (\nabla_{\partial_m} T)(dx^{i_1}, \dots, dx^{i_r}, \partial_{j_1}, \dots, \partial_{j_s}) \\ &= \partial_m T_{j_1 \dots j_s}^{i_1 \dots i_r} + T_{j_1 \dots j_s}^{h \dots i_r} \Gamma_{hm}^{i_1} + \dots + T_{j_1 \dots j_s}^{i_1 \dots h} \Gamma_{hm}^{i_r} \\ &\quad - T_{h \dots j_s}^{i_1 \dots i_r} \Gamma_{j_1 m}^h - \dots - T_{j_1 \dots h}^{i_1 \dots i_r} \Gamma_{j_s m}^h \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} L_X T_{j_1 \dots j_s}^{i_1 \dots i_r} &:= (L_X T)_{j_1 \dots j_s}^{i_1 \dots i_r} := (L_X T)(dx^{i_1}, \dots, dx^{i_r}, \partial_{j_1}, \dots, \partial_{j_s}) \\ &= X^m \partial_m T_{j_1 \dots j_s}^{i_1 \dots i_r} - T_{j_1 \dots j_s}^{m \dots i_r} (\partial_m X^{i_1}) + \dots - T_{j_1 \dots j_s}^{i_1 \dots m} (\partial_m X^{i_r}) \\ &\quad + T_{m \dots j_s}^{i_1 \dots i_r} (\partial_{j_1} X^m) - \dots - T_{j_1 \dots m}^{i_1 \dots i_r} (\partial_{j_s} X^m). \end{aligned} \tag{3.2}$$

Proposition 3.1. *Let (M, g, ∇) be a statistical manifold. Then we have the following formulas:*

$$(L_X g)(Y, Z) = -2g(K_X Y, Z) + g(\nabla_Y X, Z) + g(Y, \nabla_Z X), \tag{3.3}$$

$$(L_X g)(Y, Z) = 2g(K_X Y, Z) + g(\overset{*}{\nabla}_Y X, Z) + g(Y, \overset{*}{\nabla}_Z X). \tag{3.4}$$

Proof. Since ∇ is symmetric and $C(X, Y, Z) = -2g(K_X Y, Z)$, we have

$$\begin{aligned} (L_X g)(Y, Z) &= Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) \\ &= Xg(Y, Z) - g(\nabla_X Y, Z) + g(\nabla_Y X, Z) - g(Y, \nabla_X Z) + g(Y, \nabla_Z X) \\ &= C(X, Y, Z) + g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \\ &= -2g(K_X Y, Z) + g(\nabla_Y X, Z) + g(Y, \nabla_Z X), \end{aligned}$$

which gives us (3.3). Since $\overset{*}{\nabla} = \nabla - 2K$, we get

$$\begin{aligned} (L_X g)(Y, Z) &= Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) \\ &= Xg(Y, Z) - g(\overset{*}{\nabla}_X Y, Z) + g(\overset{*}{\nabla}_Y X, Z) - g(Y, \overset{*}{\nabla}_X Z) + g(Y, \overset{*}{\nabla}_Z X) \\ &= (\nabla_X g)(Y, Z) + 2g(K_X Y, Z) + 2g(Y, K_X Z) \\ &= 2C(X, Y, Z) + 2g(K_X Y, Z) + 2g(Y, K_X Z). \end{aligned}$$

Relations (2.3) and the above equation imply (3.4). □

Proposition 3.2. *Let (M, g, ∇) be a statistical manifold. Then we have*

$$(L_X \nabla)(Y, Z) = (L_X \overset{*}{\nabla})(Y, Z) + 2(L_X K)(Y, Z), \tag{3.5}$$

$$(L_X \nabla^{(0)})(Y, Z) = (L_X \overset{*}{\nabla})(Y, Z) + (L_X K)(Y, Z), \tag{3.6}$$

$$(L_X \nabla^{(0)})(Y, Z) = 2(L_X \overset{*}{\nabla})(Y, Z) - (L_X \nabla)(Y, Z) + 3(L_X K)(Y, Z), \tag{3.7}$$

where $(L_X \nabla)(Y, Z) = L_X(\nabla_Y Z) - \nabla_{L_X Y} Z - \nabla_Y L_X Z$.

Proof. Using (2.2) and the relations $\overset{*}{\nabla} = \nabla - 2K$ and $\nabla^{(0)} = \frac{1}{2}(\nabla + \overset{*}{\nabla})$ we can conclude (3.5)–(3.7). □

Here we study some properties of the Lie derivative of tensors on statistical manifolds in the local format. Note that the local expression of the tensors C and K introduced by (2.1) and (2.2) are $C = C_{ijk}dx^i \otimes dx^j \otimes dx^k$ and $K = K_{ij}^k dx^i \otimes dx^j \otimes \partial_k$. From (2.3) we conclude that the components K_{ij}^k of K satisfy the relations

$$K_{ij}^k = K_{ji}^k, \quad K_{ij}^r g_{rk} = K_{ik}^r g_{rj}, \quad C_{ijk} = -2K_{ij}^r g_{rk}. \tag{3.8}$$

From the above equations we conclude that C_{ijk} is totally symmetric, i.e., $C_{ijk} = C_{jik} = C_{ikj}$.

Proposition 3.3. *Let $X = X^h \partial_h$ be a vector field on a statistical manifold (M, g, ∇) . Then we have*

$$L_X g_{ij} = 2X^r K_{rjk} + \nabla_j X_i + \nabla_i X_j,$$

where $X_i = g_{ir} X^r$ and $K_{rjk} = K_{rj}^h g_{hk}$.

Proof. Setting $X = X^i \partial_i$, $Y = \partial_j$ and $Z = \partial_k$ in (3.3) we get

$$\begin{aligned} L_X g_{jk} &= -2g(K_{X^i \partial_i} \partial_j, \partial_k) + g(\nabla_{\partial_j} (X^i \partial_i), \partial_k) + g(\partial_j, \nabla_{\partial_k} (X^i \partial_i)) \\ &= -2X^i K_{ij}^r g_{rk} + (\partial_j X^i) g_{ik} + X^i \Gamma_{ji}^r g_{rk} + (\partial_k X^i) g_{ji} + X^i \Gamma_{ki}^r g_{jr} \\ &= -2X^i K_{ijk} + (\partial_j X^r + X^i \Gamma_{ji}^r) g_{rk} + (\partial_k X^r + X^i \Gamma_{ki}^r) g_{jr}. \end{aligned}$$

Using (3.1) in the above equation implies

$$L_X g_{jk} = -2X^i K_{ijk} + (\nabla_j X^r) g_{rk} + (\nabla_k X^r) g_{jr}. \tag{3.9}$$

But using (2.1) we get

$$(\nabla_j X^r) g_{rk} = \nabla_j (X^r g_{rk}) - X^r \nabla_j g_{rk} = \nabla_j X_k - X^r C_{jrk}. \tag{3.10}$$

Similarly, we have

$$(\nabla_k X^r) g_{jr} = \nabla_k X_j - X^r C_{krj}. \tag{3.11}$$

Putting (3.10) and (3.11) in (3.9) and considering that C_{jrk} is totally symmetric we obtain

$$L_X g_{jk} = -2X^i K_{ijk} + \nabla_j X_k + \nabla_k X_j - 2X^r C_{jrk}.$$

Using the third equation of (3.8) in the above equation implies

$$\begin{aligned} L_X g_{jk} &= -2X^i K_{ijk} + \nabla_j X_k + \nabla_k X_j + 4X^r K_{jrk} \\ &= 2X^r K_{rjk} + \nabla_j X_k + \nabla_k X_j. \end{aligned} \quad \square$$

Proposition 3.4. *Let (M, ∇, g) be a statistical manifold. Then we have the following formula:*

$$L_X \Gamma_{ij}^h = \nabla_i \nabla_j X^h + X^k R_{kij}^h + 2X^k (\partial_k (K_{ij}^h) - \partial_i (K_{jk}^h) - 2K_{jk}^t K_{it}^h + 2K_{ij}^t K_{tk}^h),$$

where $L_X \Gamma_{ij}^h := (L_X \nabla)_{ij}^h$.

Proof. Equation (3.5) has the local expression

$$\begin{aligned}
 L_X \Gamma_{ij}^h &= \partial_i \partial_j (X^h) + X^t \partial_t (\Gamma_{ji}^{*h}) - \partial_t (X^h) \Gamma_{ji}^{*t} + \partial_j (X^t) \Gamma_{it}^{*h} + \partial_i (X^t) \Gamma_{jt}^{*h} \\
 &\quad + 2\partial_t (K_{ji}^h) X^t - 2\partial_t (X^h) K_{ji}^t + 2\partial_j (X^t) K_{ti}^h + 2\partial_i (X^t) K_{jt}^h.
 \end{aligned}
 \tag{3.12}$$

Since ∇ is a statistical connection, we get

$$\begin{aligned}
 \nabla_i \nabla_j X^h &= \partial_i (\nabla_j X^h) + (\nabla_j X^t) \Gamma_{it}^h - (\nabla_t X^h) \gamma_{ij}^t \\
 &= \partial_i (\partial_j (X^h) + \Gamma_{jt}^h X^t) + (\partial_j (X^t) + \Gamma_{jk}^t X^k) \Gamma_{it}^h - (\partial_t (X^h) + \Gamma_{tk}^h X^k) \Gamma_{ij}^t \\
 &= \partial_i \partial_j (X^h) + \partial_i (\Gamma_{jt}^h) X^t + \Gamma_{jt}^h \partial_i (X^t) + \partial_j (X^t) \Gamma_{it}^h + X^k \Gamma_{jk}^t \Gamma_{it}^h \\
 &\quad - \partial_t (X^h) \Gamma_{ij}^t - X^k \Gamma_{tk}^h \Gamma_{ij}^t.
 \end{aligned}$$

Using $\nabla = \overset{*}{\nabla} + 2K$, the above equation reduces to

$$\begin{aligned}
 \nabla_i \nabla_j X^h &= \partial_i \partial_j (X^h) + \partial_i (\Gamma_{jt}^{*h}) X^t + \Gamma_{jt}^{*h} \partial_i (X^t) + \partial_j (X^t) \Gamma_{it}^{*h} + X^k \Gamma_{jk}^{*t} \Gamma_{it}^{*h} \\
 &\quad - \partial_t (X^h) \Gamma_{ij}^{*t} - X^k \Gamma_{tk}^{*h} \Gamma_{ij}^{*t} + 2\partial_i (K_{jt}^h) X^t + 2K_{jt}^h \partial_i (X^t) + 2\partial_j (X^t) K_{it}^h \\
 &\quad + 4X^k K_{jk}^t K_{it}^h - 2\partial_t (X^h) K_{ij}^t - 4X^k K_{tk}^h K_{ij}^t.
 \end{aligned}$$

The expression (3.12) and the above equations imply

$$\begin{aligned}
 \nabla_i \nabla_j X^h &= L_X \Gamma_{ij}^h - X^k \left(\partial_k (\Gamma_{ji}^{*h}) - \partial_i (\Gamma_{jk}^{*h}) - \Gamma_{jk}^{*t} \Gamma_{it}^{*h} + \Gamma_{tk}^{*h} \Gamma_{ij}^{*t} \right) \\
 &\quad + 2X^k \left(\partial_i (K_{jk}^h) - \partial_k (K_{ji}^h) + 2K_{jk}^t K_{it}^h - 2K_{tk}^h K_{ij}^t \right).
 \end{aligned}$$

Using the local expression of $\overset{*}{R}$ (see (2.6)) we obtain

$$\nabla_i \nabla_j X^h = L_X \Gamma_{ij}^h - X^k R_{kij}^{*h} + 2X^k (\partial_i (K_{jk}^h) - \partial_k (K_{ji}^h) + 2K_{jk}^t K_{it}^h - 2K_{tk}^h K_{ij}^t).$$

□

4. CONFORMAL VECTOR FIELDS ON STATISTICAL MANIFOLDS

In this section we present the definition of a conformal vector field on a statistical manifold and we study some examples.

It is known that the concept of conformal vector field is independent of the choice of linear connection. So, its introduction on statistical manifolds is similar to that on Riemannian ones. This motivates us to introduce a new concept that uses the statistical connection structure and gives us the definition of conformal vector field in the Riemannian case. Since the skewness operator K has a basic role in statistical manifolds, we need to consider a certain condition on it. It is known that the geometrical symmetries of spacetime (which have many applications in general relativity) are often defined through the vanishing of the Lie derivative of certain tensors with respect to a vector (see [26] for more details). So, we are interested in skewness operators whose Lie derivative is zero.

Definition 4.1. Let (g, ∇) be a statistical structure on M . A vector field X is called a *conformal vector field* if

$$L_X g = 2\rho g, \quad L_X K = 0,$$

where ρ is a function on M .

According to (3.2), the above equations have local expressions

$$L_X g_{ij} = X^r \partial_r g_{ij} + \partial_j(X^r)g_{ir} + \partial_i(X^r)g_{jr}, \tag{4.1}$$

$$L_X K_{ij}^r = X^l \partial_l K_{ij}^r - K_{ij}^l \partial_l(X^r) + \partial_i(X^l)K_{lj}^r + \partial_j(X^l)K_{li}^r. \tag{4.2}$$

Using the relations

$$\begin{aligned} (\nabla_i X^r) &= \partial_i(X^r) + X^h \Gamma_{hi}^r, \\ \nabla_r K_{ij}^m &= \partial_r K_{ij}^m + K_{ij}^h \Gamma_{hr}^m - K_{ih}^m \Gamma_{rj}^h - K_{hj}^m \Gamma_{ir}^h, \end{aligned}$$

we can rewrite (4.2) as follows

$$L_X K_{ij}^r = X^l \nabla_l K_{ij}^r - K_{ij}^l (\nabla_l X^r) + K_{lj}^r (\nabla_i X^l) + K_{li}^r (\nabla_j X^l). \tag{4.3}$$

From Definition 4.1 and equations (4.1) and (4.2) we can conclude the following:

Lemma 4.2. *A vector field $X = X^i \partial_i$ on a statistical manifold (M, g, ∇) is conformal if and only if*

$$X^r \partial_r g_{ij} + \partial_j(X^r)g_{ir} + \partial_i(X^r)g_{jr} = 2\rho g_{ij}, \tag{4.4}$$

$$X^l \partial_l K_{ij}^r - K_{ij}^l \partial_l(X^r) + \partial_i(X^l)K_{lj}^r + \partial_j(X^l)K_{li}^r = 0. \tag{4.5}$$

Example 4.3. Consider the Fisher metric

$$g = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{bmatrix}$$

on the normal distribution manifold

$$M = \left\{ f(x; \mu, \sigma) \mid f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x, \mu \in \mathbb{R}, \sigma > 0 \right\}.$$

The equations (3.8) imply

$$K_{12}^1 = 2K_{11}^2, \quad K_{22}^1 = 2K_{21}^2. \tag{4.6}$$

Let $X = X^1 \partial_1 + X^2 \partial_2$ be a vector field on M , where $\partial_1 = \frac{\partial}{\partial \mu}$ and $\partial_2 = \frac{\partial}{\partial \sigma}$. Using (4.4), we get

$$X^1 \partial_1(g_{ij}) + X^2 \partial_2(g_{ij}) + \partial_i(X^1)g_{1j} + \partial_i(X^2)g_{2j} + \partial_j(X^1)g_{i1} + \partial_j(X^2)g_{i2} = 2\rho g_{ij}.$$

Considering $i, j = 1, 2$, the above equation implies

$$-\frac{1}{\sigma} X^2 + \partial_1(X^1) = \rho, \quad -\frac{1}{\sigma} X^2 + \partial_2(X^2) = \rho, \quad \partial_2(X^1) = -2\partial_1(X^2). \tag{4.7}$$

Note that the first two equations of (4.7) imply $\partial_1(X^1) = \partial_2(X^2)$. Considering $i, j, r = 1, 2$ in (4.5) we obtain the following equations:

$$X^1\partial_1(K_{11}^1) + X^2\partial_2(K_{11}^1) + \partial_1(X^1)K_{11}^1 + 6\partial_1(X^2)K_{11}^2 = 0, \tag{4.8}$$

$$X^1\partial_1(K_{11}^2) + X^2\partial_2(K_{11}^2) - \partial_1(X^2)K_{11}^1 - \partial_2(X^2)K_{11}^2 + 2\partial_1(X^1)K_{11}^2 + 2\partial_1(X^2)K_{21}^2 = 0, \tag{4.9}$$

$$X^1\partial_1(K_{22}^1) + X^2\partial_2(K_{22}^1) - \partial_2(X^1)K_{22}^2 + 2\partial_2(X^1)K_{12}^1 + \partial_2(X^2)K_{22}^1 = 0, \tag{4.10}$$

$$X^1\partial_1(K_{22}^2) + X^2\partial_2(K_{22}^2) - \partial_1(X^2)K_{22}^1 + 2\partial_2(X^1)K_{12}^2 + \partial_2(X^2)K_{22}^2 = 0, \tag{4.11}$$

$$X^1\partial_1(K_{12}^1) + X^2\partial_2(K_{12}^1) + 4\partial_1(X^2)K_{21}^2 + \partial_2(X^1)K_{11}^1 + \partial_2(X^2)K_{12}^1 = 0, \tag{4.12}$$

$$X^1\partial_1(K_{12}^2) + X^2\partial_2(K_{12}^2) - 4\partial_1(X^2)K_{11}^2 + \partial_2(X^2)K_{12}^2 + \partial_1(X^2)K_{22}^2 = 0, \tag{4.13}$$

$$X^1\partial_1(K_{12}^1) + X^2\partial_2(K_{12}^1) + 4\partial_1(X^2)K_{21}^2 + \partial_2(X^1)K_{11}^1 + \partial_2(X^2)K_{12}^1 = 0, \tag{4.14}$$

$$X^1\partial_1(K_{12}^2) + X^2\partial_2(K_{12}^2) - 4\partial_1(X^2)K_{11}^2 + \partial_1(X^1)K_{12}^2 + \partial_1(X^2)K_{22}^2 = 0. \tag{4.15}$$

So, X is conformal if and only if it satisfies (4.6), (4.7) and (4.8)–(4.15). For instance, if we consider

$$X^1 = a\mu, \quad X^2 = a\sigma, \quad \rho = 0,$$

and

$$K_{12}^1 = 2K_{11}^2 = 0, \quad K_{11}^1 = K_{22}^1 = K_{22}^2 = 2K_{21}^2 = \frac{1}{a\sigma}, \tag{4.16}$$

where a is non-zero constant, then we obtain a conformal vector field on $(\mathbb{R}^2, g, \nabla)$. Note that the coefficients of the Levi-Civita connection $\nabla^{(0)}$ of the Fisher metric given by (4.3) are

$$\Gamma_{12}^{(0)1} = \Gamma_{21}^{(0)1} = \Gamma_{22}^{(0)2} = -2\Gamma_{11}^{(0)2} = -\frac{1}{\sigma}, \quad \Gamma_{11}^{(0)1} = \Gamma_{12}^{(0)2} = \Gamma_{21}^{(0)2} = \Gamma_{22}^{(0)1} = 0;$$

then using (2.2) and (4.16) we deduce that the coefficients of the statistical connection ∇ are as follows:

$$\Gamma_{11}^1 = 2\Gamma_{12}^2 = 2\Gamma_{21}^2 = \Gamma_{22}^1 = \frac{1}{a\sigma}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = -2\Gamma_{11}^2 = -\frac{1}{\sigma}, \quad \Gamma_{22}^2 = \frac{1-a}{a\sigma}.$$

Example 4.4. We consider the normal distribution Riemannian manifold (M, g) introduced in Example 4.3. Let η be a 1-form on M . It is easy to see that

$$K_X Y = g(X, Y)\eta^\sharp + \eta(X)Y + \eta(Y)X,$$

where $\eta(X) = g(X, \eta^\sharp)$ satisfies (i) and (ii) of (2.3). So, the linear connection ∇ given by

$$\nabla_X Y = \nabla_X^{(0)} Y + g(X, Y)\eta^\sharp + \eta(X)Y + \eta(Y)X$$

is a statistical connection on (M, g) (this connection has been introduced by Blaga and Crasmareanu [10]). Now, let $X = X^1\partial_1 + X^2\partial_2$ be a vector field on M . Using (4.3), we get

$$L_X K_{ij}^r = X^l \partial_l (g_{ij} \eta^{\sharp r} + \eta_i \delta_j^r + \eta_j \delta_i^r) - \partial_m (X^r) (g_{ij} \eta^{\sharp m} + \eta_i \delta_j^m + \eta_j \delta_i^m) + \partial_i (X^l) (g_{lj} \eta^{\sharp r} + \eta_l \delta_j^r + \eta_j \delta_l^r) + \partial_j (X^l) (g_{li} \eta^{\sharp r} + \eta_l \delta_i^r + \eta_i \delta_l^r),$$

where $K_{ij}^r = g_{ij}\eta^{#r} + \eta_i\delta_j^r + \eta_j\delta_i^r$. Considering $i, j = 1, 2$ and using $\eta^{#i} = g^{ij}\eta_j$, the above equation induces the following:

$$\begin{aligned} 3X^1\partial_1(\eta_1) + 3X^2\partial_2(\eta_1) - \frac{1}{2}\partial_2(X^1)\eta_2 + 3\partial_1(X^1)\eta_1 + 2\partial_1(X^2)\eta_2 &= 0, \\ \frac{1}{2}X^1\partial_1(\eta_2) + \frac{1}{2}X^2\partial_2(\eta_2) - \partial_1(X^2)\eta_1 - \frac{1}{2}\partial_2(X^2)\eta_2 + \partial_1(X^1)\eta_2 &= 0, \\ 2X^1\partial_1(\eta_1) + 2X^2\partial_2(\eta_1) - 2\partial_1(X^1)\eta_1 - \partial_2(X^1)\eta_2 + 4\partial_2(X^2)\eta_1 &= 0, \\ 3X^1\partial_1(\eta_2) + 3X^2\partial_2(\eta_2) - 2\partial_1(X^2)\eta_1 + 2\partial_2(X^1)\eta_1 + 3\partial_2(X^2)\eta_2 &= 0, \\ X^1\partial_1(\eta_2) + X^2\partial_2(\eta_2) + 2\partial_1(X^2)\eta_1 + 2\partial_2(X^1)\eta_1 + \partial_2(X^2)\eta_2 &= 0, \\ X^1\partial_1(\eta_1) + X^2\partial_2(\eta_1) + \partial_1(X^1)\eta_1 + 2\partial_1(X^2)\eta_2 + \frac{1}{2}\partial_2(X^1)\eta_2 &= 0. \end{aligned}$$

Using (4.7), the six equations above reduce to

$$\begin{aligned} X^1\partial_1(\eta_1) + X^2\partial_2(\eta_1) + \partial_1(X^2)\eta_2 + \partial_1(X^1)\eta_1 &= 0, \\ X^1\partial_1(\eta_2) + X^2\partial_2(\eta_2) - 2\partial_1(X^2)\eta_1 + \partial_1(X^1)\eta_2 &= 0. \end{aligned}$$

So, X is conformal if and only if it satisfies these two equations. For instance, if we consider $\eta = \frac{\lambda}{k\mu+c}d\mu + \frac{\lambda}{k\mu+c}d\sigma$, where λ, k and c are constants, then $X = (k\mu + c)\partial_1 + (k\sigma + c)\partial_2$ is a conformal vector field on $(\mathbb{R}^2, g, \nabla)$ with $\rho = -\frac{c}{\sigma}$.

Here we study the conformal geometry on two famous statistical manifolds. One of them is the generalized Gaussian distribution manifold and the other is a p -dimensional manifold (of course for $p = 2$). The geometric structures of these manifolds were studied in [34].

The *generalized Gaussian distribution manifold* is defined as

$$M_1 = \left\{ f(x; \mu, \sigma, \beta) \mid f(x; \mu, \sigma, \beta) = \frac{\beta}{2\sigma\Gamma(\frac{1}{\beta})} e^{-\frac{|x-\mu|^\beta}{\sigma^\beta}}, x, \mu \in \mathbb{R}, \sigma, \beta > 0 \right\},$$

where $\Gamma(x)$ is the gamma function and μ, σ and β are called the *location, scale and shape parameters*, respectively. When $\beta = 1$ or $\beta = 2$, this distribution reduces to the Laplace distribution or the Gaussian distribution, respectively. Note that if β is odd, the manifold is not smooth. Hence we only consider the case when β is a known even number. In [34], Yuan proved that the Riemannian metric on the generalized Gaussian statistical manifold M_1 is as follows:

$$g = \begin{bmatrix} \frac{1}{\sigma^2}c_{11} & 0 \\ 0 & \frac{1}{\sigma^2}c_{22} \end{bmatrix}, \tag{4.17}$$

where

$$c_{11} = \frac{\Gamma(1 - \frac{1}{\beta})\beta(\beta - 1)}{\Gamma(\frac{1}{\beta})}, \quad c_{22} = \beta.$$

Also, g^{-1} , given by

$$g^{-1} = \begin{bmatrix} \sigma^2 & 0 \\ c_{11} & \sigma^2 \\ 0 & c_{22} \end{bmatrix},$$

is the inverse of g . The Christoffel symbols of the α -connection are as follows:

$$\begin{cases} \Gamma_{11}^{(\alpha)1} = \Gamma_{12}^{(\alpha)2} = \Gamma_{21}^{(\alpha)2} = \Gamma_{22}^{(\alpha)1} = 0, & \Gamma_{22}^{(\alpha)2} = \left(c_{222}^{(1)} + \frac{1-\alpha}{2} c_{222} \right) \frac{1}{\sigma c_{22}}, \\ \Gamma_{21}^{(\alpha)1} = \Gamma_{12}^{(\alpha)1} = \left(c_{121}^{(1)} + \frac{1-\alpha}{2} c_{121} \right) \frac{1}{\sigma c_{11}}, & \Gamma_{11}^{(\alpha)2} = \left(c_{112}^{(1)} + \frac{1-\alpha}{2} c_{112} \right) \frac{1}{\sigma c_{22}}, \end{cases} \tag{4.18}$$

where

$$\begin{cases} c_{121} = c_{112} = \frac{\Gamma\left(\frac{3\beta-1}{\beta}\right)\beta^3 - \Gamma\left(\frac{2\beta-1}{\beta}\right)\beta^2}{\Gamma\left(\frac{1}{\beta}\right)}, \\ c_{222} = 2\beta^2, \\ c_{112}^{(1)} = \frac{\Gamma\left(\frac{\beta-1}{\beta}\right)\beta(\beta-1) - \Gamma\left(\frac{2\beta-1}{\beta}\right)\beta^2(\beta-1)}{\Gamma\left(\frac{1}{\beta}\right)}, \\ c_{121}^{(1)} = -\frac{\Gamma\left(\frac{2\beta-1}{\beta}\right)\beta^3}{\Gamma\left(\frac{1}{\beta}\right)}, \\ c_{222}^{(1)} = \beta(1-\beta). \end{cases}$$

Considering $\alpha = 0$, we can obtain the Christoffel symbols of the Levi-Civita connection as follows:

$$\begin{cases} \Gamma_{11}^{(0)1} = \Gamma_{12}^{(0)2} = \Gamma_{21}^{(0)2} = \Gamma_{22}^{(0)1} = 0, & \Gamma_{22}^{(0)2} = \left(c_{222}^{(1)} + \frac{1}{2} c_{222} \right) \frac{1}{\sigma c_{22}}, \\ \Gamma_{21}^{(0)1} = \Gamma_{12}^{(0)1} = \left(c_{121}^{(1)} + \frac{1}{2} c_{121} \right) \frac{1}{\sigma c_{11}}, & \Gamma_{11}^{(0)2} = \left(c_{112}^{(1)} + \frac{1}{2} c_{112} \right) \frac{1}{\sigma c_{22}}. \end{cases} \tag{4.19}$$

Theorem 4.5. *Every conformal vector field on the generalized Gaussian distribution manifold M_1 is Killing. Moreover, $X = X^1\partial_1 + X^2\partial_2$ is a Killing vector field if and only if $X^1 = A\mu + B$ and $X^2 = A\sigma$, where A and B are constants, $\partial_1 = \frac{\partial}{\partial\mu}$ and $\partial_2 = \frac{\partial}{\partial\sigma}$.*

Proof. Using $K_{ij}^r = \Gamma_{ij}^{(\alpha)r} - \Gamma_{ij}^{(0)r}$, (4.18) and (4.19) we get

$$\begin{aligned} K_{11}^1 &= K_{12}^2 = K_{21}^2 = K_{22}^1 = 0, & K_{11}^2 &= \frac{-\alpha}{2\sigma c_{22}} c_{112}, \\ K_{22}^2 &= \frac{-\alpha}{2\sigma c_{22}} c_{222}, & K_{21}^1 &= K_{12}^1 = \frac{-\alpha}{2\sigma c_{11}} c_{121}. \end{aligned} \tag{4.20}$$

Let $X = X^1\partial_1 + X^2\partial_2$ be a conformal vector field on M_1 . Considering $i, j = 1, 2$ in (4.4) we obtain

$$\frac{-1}{\sigma} X^2 + \partial_1(X^1) = \rho, \quad \frac{-1}{\sigma} X^2 + \partial_2(X^2) = \rho, \quad \partial_2(X^1)c_{11} + \partial_1(X^2)c_{22} = 0.$$

The above equations imply

$$\partial_1(X^1) = \partial_2(X^2) = \rho + \frac{1}{\sigma} X^2, \quad \partial_2(X^1) = -\frac{c_{22}}{c_{11}} \partial_1(X^2). \tag{4.21}$$

Also, putting $i, j, r = 1, 2$ in (4.5), we obtain the following equations:

$$\begin{aligned}
 & \text{(i) } 2K_{21}^1 \partial_1(X^2) - K_{11}^2 \partial_2(X^1) = 0, \\
 & \text{(ii) } X^2 \partial_2(K_{22}^2) + K_{22}^2 \partial_2(X^2) = 0, \\
 & \text{(iii) } X^2 \partial_2(K_{21}^1) + K_{21}^1 \partial_2(X^2) = 0, \\
 & \text{(iv) } X^2 \partial_2(K_{11}^2) - K_{11}^2 \partial_2(X^2) + 2K_{11}^2 \partial_1(X^1) = 0, \\
 & \text{(v) } 2K_{12}^1 \partial_2(X^1) - K_{22}^2 \partial_2(X^1) = 0, \\
 & \text{(vi) } K_{22}^2 \partial_1(X^2) - K_{12}^1 \partial_1(X^2) + K_{11}^2 \partial_2(X^1) = 0.
 \end{aligned} \tag{4.22}$$

The equations (4.20) and (4.22) (ii) give us $\frac{\alpha}{2\sigma c_{22}} c_{222} (\frac{1}{\sigma} X^2 - \partial_2(X^2)) = 0$. Since $c_{222} \neq 0$, we conclude that $\partial_2(X^2) = \frac{1}{\sigma} X^2$. Setting this equation in the first equation of (4.21) we get $\rho = 0$. So X reduces to a Killing vector field. Putting the second equation of (4.21) in (4.22) (i) yields $(2K_{21}^1 + K_{11}^2 \frac{c_{22}}{c_{11}}) \partial_1(X^2) = 0$. Considering (4.20), this equation reduces to $-\frac{3\alpha}{\sigma c_{11}} c_{112} \partial_1(X^2) = 0$. In a similar way, (4.22) (vi) reduces to $\frac{\alpha}{2\sigma c_{11} c_{22}} (2c_{22} c_{112} - c_{11} c_{222}) \partial_1(X^2) = 0$. If $\partial_1(X^2) \neq 0$, then the last two equations imply $c_{222} = 0$, which is a contradiction. Thus $\partial_1(X^2) = 0$. Finally, considering (4.21), we get $\partial_2(X^1) = 0$ and $\partial_1(X^1) = \frac{1}{\sigma} X^2$. The differential equation system

$$\partial_1(X^1) = \partial_2(X^2) = \frac{1}{\sigma} X^2, \quad \partial_1(X^2) = \partial_2(X^1) = 0$$

has the solution $X^2 = A\sigma$ and $X^1 = A\mu + B$. □

Here we consider a p -dimensional statistical manifold. The importance of this distribution family lies in that its member is a non-Gaussian multivariate distribution, while the marginal distribution is Gaussian, which implies that a set of marginal distributions does not uniquely determine the multivariate normal distribution [14]. A p -dimensional statistical manifold is defined by

$$M_2 = \left\{ f(\mathbf{x}; \lambda) \mid f(\mathbf{x}; \lambda) = 2 \prod_{i=1}^p \frac{\sqrt{\lambda_i}}{\sqrt{2\pi}} e^{-\frac{\lambda_i x_i^2}{2}}, \mathbf{x} \in \Omega_p, \lambda \in \mathbb{R}_+^p \right\},$$

where

$$\begin{aligned}
 \Omega_p &= \{x = (x_1, \dots, x_p) \in \mathbb{R}^p \mid \prod_{i=1}^p x_i > 0\}, \\
 \mathbb{R}_+^p &= \{x = (x_1, \dots, x_p) \in \mathbb{R}^p \mid x_i > 0, i = 1, \dots, p\}.
 \end{aligned}$$

The distribution in M_2 can be rewritten as

$$f(\mathbf{x}; \lambda) = e^{\frac{1}{2} \sum_{i=1}^p \log(-\theta_i)} + \sum_{i=1}^p \theta_i x_i^2 + \frac{p}{2} \log 2 - \log \sqrt{2\pi},$$

where $\theta_i = -\frac{1}{2} \lambda_i$. This is one member of the exponential family with the natural coordinates $(\theta_1, \dots, \theta_p)$ and the potential function $\psi(\theta) = -\frac{1}{2} \sum_{i=1}^p \log(-\theta_i)$. It is known that, for the exponential family, the Fisher information is just the second

derivative of the potential function,

$$g_{ij} = \frac{\partial^2 \psi}{\partial \theta_i \partial \theta_j} = -\frac{1}{2} \frac{1}{\theta_i \theta_j} \delta_{ij}, \tag{4.23}$$

and the α -connection is the third derivative of the potential function,

$$\Gamma_{ijk}^{(\alpha)} = \frac{1-\alpha}{2} \frac{\partial^3 \psi}{\partial \theta_i \partial \theta_j \partial \theta_k} = -\frac{1-\alpha}{2} \frac{1}{\theta_i \theta_j \theta_k} \delta_{ijk}, \tag{4.24}$$

where $\delta_{ii} = 1$ for $i = 1, \dots, p$, $\delta_{ij} = 0$ for $i \neq j$, $\delta_{iii} = 1$ for $i = 1, \dots, p$, and $\delta_{ijk} = 0$ for unequal i, j, k (see [34] for more details). For $p = 2$, the matrix expression of the metric g given by (4.23) and its inverse matrix are as follows:

$$g = \begin{bmatrix} -\frac{1}{2\theta_1^2} & 0 \\ 0 & -\frac{1}{2\theta_2^2} \end{bmatrix}, \quad g^{-1} = \begin{bmatrix} -2\theta_1^2 & 0 \\ 0 & -2\theta_2^2 \end{bmatrix}. \tag{4.25}$$

From (4.24) and (4.23), we get

$$\Gamma_{11}^{(\alpha)1} = \frac{1-\alpha}{\theta_1}, \quad \Gamma_{22}^{(\alpha)2} = \frac{1-\alpha}{\theta_2}, \quad \Gamma_{ij}^{(\alpha)k} = 0 \quad \text{for unequal } i, j, k. \tag{4.26}$$

Theorem 4.6. *Every conformal vector field on the 2-dimensional statistical manifold M_2 is Killing. Moreover, $X = X^1 \partial_1 + X^2 \partial_2$ is a Killing vector field if and only if $X^1 = A\theta_1$ and $X^2 = B\theta_2$, where A and B are constants, $\partial_1 = \frac{\partial}{\partial \theta_1}$ and $\partial_2 = \frac{\partial}{\partial \theta_2}$.*

Proof. From (4.26) we have the Christoffel symbols of the Levi-Civita connection as follows:

$$\Gamma_{11}^{(0)1} = -\frac{1}{\theta_1}, \quad \Gamma_{22}^{(0)2} = -\frac{1}{\theta_2}, \quad \Gamma_{ij}^{(0)k} = 0 \quad \text{for unequal } i, j, k. \tag{4.27}$$

Using $K_{ij}^r = \Gamma_{ij}^{(\alpha)r} - \Gamma_{ij}^{(0)r}$, (4.26) and (4.27) we get

$$K_{11}^1 = \frac{\alpha}{\theta_1}, \quad K_{22}^2 = \frac{\alpha}{\theta_2}, \quad K_{ij}^k = 0 \quad \text{for unequal } i, j, k.$$

Now let $X = X^1 \partial_1 + X^2 \partial_2$ be a conformal vector field on M_2 . By (4.4) we obtain

$$\partial_1(X^1) - \frac{X^1}{\theta_1} = \rho, \quad \partial_2(X^2) - \frac{X^2}{\theta_2} = \rho, \quad \frac{\partial_2(X^1)}{\theta_1^2} + \frac{\partial_1(X^2)}{\theta_2^2} = 0. \tag{4.28}$$

Setting $i = j = r = 1$ in (4.5) implies $\partial_1(X^1) = \frac{X^1}{\theta_1}$. Considering this equation and the first equation of (4.28) we conclude that $\rho = 0$. So X is a Killing vector field. Putting $i = j = 1, r = 2$ in (4.5) yields $K_{11}^1 \partial_1(X^2) = 0$, which gives us $\partial_1(X^2) = 0$. This equation together with the third equation of (4.28) results in $\partial_2(X^1) = 0$. From this equation and $\partial_1(X^1) = \frac{X^1}{\theta_1}$ we get $X^1 = A\theta_1$. Similarly, the second equation of (4.28) and $\partial_2(X^1) = 0$ imply $X^2 = B\theta_2$. It is easy to see that $X = A\theta_1 \partial_1 + B\theta_2 \partial_2$ satisfies all the equations of (4.4) and (4.5). \square

5. CONFORMAL VECTOR FIELDS ON THE TANGENT BUNDLE

In this section, we consider two statistical structures on the tangent bundle of a statistical manifold and we study the conformal vector fields on these structures.

Let (M, g, ∇) be a statistical manifold with the skewness operator K . Using (2.7) and (2.9), the horizontal lift metric g^h with respect to ∇ is described by the formulas

$$g^h(X^h, Y^h) = g^h(X^v, Y^v) = 0, \quad g^h(X^h, Y^v) = g(X, Y) \tag{5.1}$$

for all $X, Y \in \chi(M)$. Again, using (2.7) and (2.9), the horizontal lift of K is defined by

$$K^h_{X^h} Y^h = (K_X Y)^h, \quad K^h_{X^v} Y^v = 0, \quad K^h_{X^h} Y^v = K^h_{X^v} Y^h = (K_X Y)^v. \tag{5.2}$$

In [23] Matsuzoe and Inoguchi proved that if (M, g, ∇) is a statistical manifold, then (TM, g^h, K^h) is a statistical manifold. Here we study the conditions under which X^h can be conformal on (TM, g^h, K^h) . According to Definition 4.1, a vector field $\tilde{X} \in \chi(TM)$ is called a conformal vector field on a statistical manifold $(TM, \tilde{g}, \tilde{\nabla})$ if there exists a function $\tilde{\rho}(x, y)$ on TM such that $L_{\tilde{X}} \tilde{g} = 2\tilde{\rho}\tilde{g}$ and $L_{\tilde{X}} \tilde{K} = 0$, where \tilde{K} is the skewness operator associated to $\tilde{\nabla}$. If $\tilde{\rho}$ is a function that depends only on x^h , then \tilde{X} is called an *inessential* vector field.

Theorem 5.1. *Let (M, g, ∇) be a statistical manifold with the skewness operator K and let X^h be the horizontal lift of a vector field on M . If X^h is conformal with respect to (TM, g^h, K^h) , then X^h is an inessential vector field. Moreover, X^h is an inessential vector field if and only if*

$$X^r (R_{rikj} + R_{rjki}) = 0, \tag{5.3}$$

$$X^r \nabla_r g_{ij} + (\nabla_i X^r) g_{rj} = 2\tilde{\rho}(x) g_{ij}, \tag{5.4}$$

$$-K^r_{ij} (\nabla_r X^m) + K^m_{ri} (\nabla_j X^r) = 0, \tag{5.5}$$

$$X^r (R^h_{rik} K^m_{hj} + R^h_{rjk} K^m_{hi}) = 0, \tag{5.6}$$

$$X^r \nabla_r K^m_{ij} + (\nabla_i X^r) K^m_{rj} = 0, \tag{5.7}$$

where $R_{rikj} := R^h_{rik} g_{hj}$.

Proof. We can rewrite the metric g^h and K^h defined by (5.1) and (5.2) as follows:

$$g^h(\delta_i, \delta_j) = g^h(\partial_i, \partial_j) = 0, \quad g^h(\delta_i, \partial_j) = g^h(\partial_i, \delta_j) = g_{ij}, \tag{5.8}$$

$$K^h_{\delta_i} \delta_j = K^h_{ij} \delta_h, \quad K^h_{\partial_i} \partial_j = 0, \quad K^h_{\delta_i} \partial_j = K^h_{\partial_i} \delta_j = K^h_{ij} \partial_h. \tag{5.9}$$

Using (2.5) and (5.8) we get $(L_{X^h} g^h)(\partial_i, \partial_j) = 0$. In a similar way, we obtain

$$(L_{X^h} g^h)(\delta_i, \delta_j) = X^r y^k (R_{rikj} + R_{rjki}), \tag{5.10}$$

$$(L_{X^h} g^h)(\delta_i, \partial_j) = X^r \partial_r g_{ij} + (\partial_i X^r) g_{rj} - X^r \Gamma^h_{rj} g_{hi} = X^r \nabla_r g_{ij} + (\nabla_i X^r) g_{rj}. \tag{5.11}$$

The relations (2.5) and (5.9) imply

$$\begin{aligned} (L_{X^h} K^h)(\partial_i, \partial_j) &= 0, \\ (L_{X^h} K^h)(\delta_i, \delta_j) &= (L_X K)_{ij}^m \delta_m + X^r y^k (R_{rik}^h K_{hj}^m + R_{rjk}^h K_{hi}^m) \partial_{\bar{m}}, \\ (L_{X^h} K^h)(\delta_i, \partial_j) &= \{X^r \partial_r (K_{ij}^m) + \partial_i (X^r) K_{rj}^m + X^r (K_{ij}^h \Gamma_{rh}^m - K_{ih}^m \Gamma_{rj}^h)\} \partial_{\bar{m}}, \\ &= \{X^r \nabla_r K_{ij}^m + (\nabla_i X^r) K_{rj}^m\} \partial_{\bar{m}}. \end{aligned}$$

The above equations and (4.3) imply that $L_{X^h} K^h = 0$ if and only if (5.5)–(5.7) hold. Using (5.11) we obtain

$$(L_{X^h} g^h)(\delta_i, \partial_j) = X^r \nabla_r g_{ij} + (\nabla_i X^r) g_{rj} = 2\tilde{\rho} g_{ij}.$$

Applying $\partial_{\bar{k}}$ to this relation gives

$$0 = 2 \frac{\partial \tilde{\rho}}{\partial y^k} g_{ij} = 2(\partial_{\bar{k}} \tilde{\rho}) g_{ij}.$$

Multiplying the last equation with g^{ij} we get

$$0 = 2n \partial_{\bar{k}} \tilde{\rho},$$

which implies that $\tilde{\rho}$ is a function with respect to x^h . So using (5.10) and (5.11) it results that $L_X^h g^h = 2\tilde{\rho} g^h$ if and only if (5.3) and (5.4) hold. \square

Example 5.2. Consider the normal distribution manifold (M, g) given by Example 4.3. It is easy to see that the Christoffel symbols of the Levi-Civita connection $\nabla^{(0)}$ of (M, g) are as follows:

$$\Gamma_{11}^{(0)1} = \Gamma_{22}^{(0)1} = \Gamma_{12}^{(0)2} = \Gamma_{21}^{(0)2} = 0, \quad \Gamma_{21}^{(0)1} = \Gamma_{12}^{(0)1} = \Gamma_{22}^{(0)2} = \frac{-1}{\sigma}, \quad \Gamma_{11}^{(0)2} = \frac{1}{2\sigma}.$$

Now we consider the tensor K with the following components:

$$K_{12}^1 = 2K_{11}^2 = 0, \quad K_{11}^1 = K_{22}^1 = K_{22}^2 = 2K_{21}^2 = \frac{1}{2a\sigma}.$$

It is easy to check that the above components satisfy (3.8). So

$$\nabla_X Y = \nabla_X^{(0)} Y + K_X Y \quad \forall X, Y \in \Gamma(TM)$$

is a statistical connection on (M, g) with the following Christoffel symbols:

$$\begin{aligned} \Gamma_{11}^1 &= \frac{1}{2a\sigma}, & \Gamma_{11}^2 &= \frac{1}{2\sigma}, & \Gamma_{22}^1 &= \frac{1}{2a\sigma}, \\ \Gamma_{22}^2 &= \frac{-2a + 1}{2a\sigma}, & \Gamma_{12}^1 &= \Gamma_{21}^1 = \frac{-1}{\sigma}, & \Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{1}{4a\sigma}. \end{aligned}$$

Using (2.6), we can show that all of the curvature components are zero except for the following ones:

$$\begin{aligned} R_{121}^2 &= -R_{211}^2 = \frac{8a^2 - 4a - 1}{16a^2\sigma^2}, & R_{122}^1 &= -R_{212}^1 = \frac{-8a^2 - 4a + 1}{8a^2\sigma^2}, \\ R_{122}^2 &= -R_{212}^2 = \frac{3}{4a\sigma^2}. \end{aligned}$$

From the above equations we deduce that

$$R_{1212} = -R_{2112} = \frac{8a^2 - 4a - 1}{8a^2\sigma^4}, \quad R_{1221} = -R_{2121} = \frac{-8a^2 - 4a + 1}{8a^2\sigma^4}, \tag{5.12}$$

$$R_{1222} = -R_{2122} = \frac{3}{2a\sigma^4}.$$

It is worth remarking that (M, g, ∇) is a non-flat statistical manifold, because $R_{1222} \neq 0$. Applying $i = j = 2$ in (5.3) and using the above equations we get

$$X^1 y^1 \left(\frac{8a^2 - 4a - 1}{8a^2\sigma^4} \right) + X^1 y^2 \left(\frac{3}{2a\sigma^4} \right) = 0,$$

where $y^1 = d\mu$ and $y^2 = d\sigma$. Differentiating the above equation with respect to y^2 implies $X^1 = 0$. Setting $i = 1$ and $j = 2$ in (5.3) and using $X^1 = 0$ and (5.12) lead to

$$X^2 y^1 \left(\frac{8a^2 - 4a - 1}{8a^2\sigma^4} \right) + X^2 y^2 \left(\frac{3}{2a\sigma^4} \right) = 0.$$

Differentiating the above equation with respect to y^2 implies $X^2 = 0$. So $X = 0$ is the only conformal vector field on the statistical manifold (M, g, ∇) such that X^h can be a conformal vector field on (TM, g^h, K^h) (with $\tilde{\rho} = 0$). Indeed, $X^h = 0$ is the only Killing vector field on (TM, g^h, K^h) .

Remark 5.3. As we can see in (5.12), $R_{1222} \neq 0$. This shows that the relation $R_{ijkl} = -R_{ijlk}$ does not hold for a non-Levi-Civita statistical connection on a Riemannian manifold.

Lemma 5.4. Consider the generalized Gaussian distribution manifold $(M_1, g, \nabla^{(\alpha)})$. M_1 is flat if and only if $\alpha = 1$ or $\alpha = \frac{1}{\beta-1}$.

Proof. In [34], Yuan proved that

$$R_{1212}^{(\alpha)} = \frac{(\alpha - 1)\beta(\beta - 1)[2 - \beta + (1 - \alpha)(\beta - 1)]\Gamma\left(\frac{\beta-1}{\beta}\right)}{\sigma^4\Gamma\left(\frac{1}{\beta}\right)}.$$

Since $\beta \neq 0, 1$, from the above equation we conclude that $R_{1212}^{(\alpha)} = 0$ if and only if $\alpha = 1$ or $2 - \beta + (1 - \alpha)(\beta - 1) = 0$. □

Theorem 5.5. Consider the generalized Gaussian distribution manifold $(M_1, g, \nabla^{(\alpha)})$. There does not exist any non-zero vector field on M_1 such that X^h is a conformal vector field with respect to (TM_1, g^h, K^h) .

Proof. Let $X = X^1\partial_1 + X^2\partial_2$ be a vector field on M_1 such that X^h is a conformal vector field with respect to (TM_1, g^h, K^h) . We consider two cases:

Case 1: $\alpha \neq 1, \frac{1}{\beta-1}$. In this case, we have $R_{1212}^{(\alpha)} \neq 0$ (see Lemma 5.4). Setting $i = j = 2$ in (5.3) implies $X^1 y^1 R_{1212}^{(\alpha)} = 0$, where $y^1 = d\mu$. Since $R_{1212}^{(\alpha)} \neq 0$, we deduce that $X^1 y^1 = 0$, which gives us $X^1 = 0$. Similarly, putting $i = 1, j = 2$ in (5.3) yields $X^2 y^1 R_{2112}^{(\alpha)} = 0$. This equation gives us $X^2 y^1 = 0$, because $R_{2112}^{(\alpha)} = -R_{1212}^{(\alpha)} \neq 0$. So we get $X^2 = 0$. Therefore, we conclude that $X = 0$.

Case 2: $\alpha = 1$ or $\alpha = \frac{1}{\beta-1}$. Setting $i = j = 2$ in (5.4) and using (4.17) we have

$$-2\frac{c_{22}}{\sigma}X^2 + c_{22}(\partial_2 X^2) - \frac{1}{\sigma}X^2 \left(c_{222}^{(1)} + \frac{1-\alpha}{2}c_{222} \right) = 2\tilde{\rho}(\mu, \sigma)c_{22}. \tag{5.13}$$

Setting $i = j = m = 1$ in (5.5) we have

$$-K_{11}^r(\nabla_r X^1) + K_{r1}^1(\nabla_1 X^r) = 0.$$

Using (4.20) in the above equation gives us

$$\frac{1}{c_{22}}(\nabla_2 X^1) - \frac{1}{c_{11}}(\nabla_1 X^2) = 0. \tag{5.14}$$

But from (3.1) and (4.18) we obtain the following:

$$\nabla_2 X^1 = \partial_2 X^1 + \frac{X^1}{\sigma c_{11}} \left(c_{121}^{(1)} + \frac{1-\alpha}{2}c_{121} \right), \tag{5.15}$$

$$\nabla_1 X^2 = \partial_1 X^2 + \frac{X^1}{\sigma c_{22}} \left(c_{112}^{(1)} + \frac{1-\alpha}{2}c_{112} \right). \tag{5.16}$$

Putting (5.15) and (5.16) in (5.14) we get

$$\frac{\partial_2 X^1}{c_{22}} - \frac{\partial_1 X^2}{c_{11}} + \frac{X^1}{\sigma c_{11}c_{22}}(c_{121}^{(1)} - c_{112}^{(1)}) = 0. \tag{5.17}$$

Setting $i = j = m = 1$ in (5.7) and using (4.20) we get

$$X^1\nabla_1 K_{11}^1 + X^2\nabla_2 K_{11}^1 + (\nabla_1 X^2)K_{21}^1 = 0. \tag{5.18}$$

Using (3.1), (4.18) and (4.20) we get

$$\nabla_1 K_{11}^1 = -2K_{21}^1\Gamma_{11}^{(\alpha)2} + K_{11}^2\Gamma_{21}^{(\alpha)1}, \quad \nabla_2 K_{11}^1 = 0, \quad \nabla_1 X^2 = \partial_1 X^2 + X^1\Gamma_{11}^{(\alpha)2}. \tag{5.19}$$

Setting (5.19) in (5.18) gives us

$$(\partial_1 X^2)K_{21}^1 + X^1K_{11}^2\Gamma_{21}^{(\alpha)1} - X^1K_{21}^1\Gamma_{11}^{(\alpha)2} = 0.$$

From (4.18), (4.20) and the above equation we deduce that (note that $c_{121} \neq 0$):

$$\frac{\partial_1 X^2}{c_{11}} + \frac{X^1}{\sigma c_{11}c_{22}}(c_{121}^{(1)} - c_{112}^{(1)}) = 0. \tag{5.20}$$

Subtracting (5.17) and (5.20) implies

$$\partial_2 X^1 = \frac{2c_{22}}{c_{11}}\partial_1 X^2. \tag{5.21}$$

Setting $i = j = m = 2$ in (5.7) and using (4.20) we obtain

$$X^1\nabla_1 K_{22}^2 + X^2\nabla_2 K_{22}^2 + (\nabla_2 X^2)K_{22}^2 = 0.$$

Using (3.1) and (4.20) the above equation reduces to

$$X^2\partial_2 K_{22}^2 + (\partial_2 X^2)K_{22}^2 = 0.$$

Setting (4.20) in the above equation yields

$$\partial_2 X^2 = \frac{X^2}{\sigma}. \tag{5.22}$$

Putting (5.22) in (5.13) we get

$$X^2 = -\frac{2c_{22}}{c_{22} + c_{222}^{(1)} + \frac{1-\alpha}{2}c_{222}}\tilde{\rho}\sigma. \tag{5.23}$$

Setting the above equation in (5.20) implies

$$X^1 = \frac{2c_{22}^2}{(c_{22} + c_{222}^{(1)} + \frac{1-\alpha}{2}c_{222})(c_{121}^{(1)} - c_{112}^{(1)})}(\partial_1\tilde{\rho})\sigma^2. \tag{5.24}$$

Using (5.23) and (5.22) we deduce that $\partial_2\tilde{\rho} = 0$, i.e., $\tilde{\rho}$ depends only on μ . So, setting (5.23) and (5.24) in (5.21) we get

$$(\partial_1\tilde{\rho})\left(\frac{1}{c_{121}^{(1)} - c_{112}^{(1)}} + \frac{1}{c_{11}}\right) = 0,$$

which implies that $\partial_1\tilde{\rho} = 0$ or $c_{11} + c_{121}^{(1)} - c_{112}^{(1)} = 0$. As $\beta \neq 0$, we get $c_{11} + c_{121}^{(1)} - c_{112}^{(1)} \neq 0$. Therefore, $\partial_1\tilde{\rho} = 0$, i.e., $\tilde{\rho}$ is constant. Considering this fact in (5.24) we get $X^1 = 0$. Setting $i = j = 1$ in (5.4) we get

$$X^2 = -\frac{2c_{11}}{2c_{11} + c_{121}^{(1)} + \frac{1-\alpha}{2}c_{121}}\tilde{\rho}\sigma. \tag{5.25}$$

The equations (5.23) and (5.25) imply $\Gamma(1 - \frac{1}{\beta})(\alpha - 1)(\beta - 1) = 0$. Since $\beta \neq 1$, the last equation gives $\alpha = 1$. Setting $i = 1, j = m = 2$ in (5.5) and using (4.18) and (4.20) we have $K_{12}^1(\Gamma_{22}^{(1)2} - \Gamma_{12}^{(1)1})X^2 = 0$, which implies that $K_{12}^1 = 0$ or $\Gamma_{22}^{(1)2} = \Gamma_{12}^{(1)1}$ or $X^2 = 0$. If $K_{12}^1 = 0$, then we get $\beta = 1$, which is a contradiction. From $\Gamma_{22}^{(1)2} = \Gamma_{12}^{(1)1}$ we get the contradiction $1 = 0$. So the possible case is $X^2 = 0$. Therefore, we conclude that $X^1 = X^2 = 0$, i.e., $X = 0$. \square

Theorem 5.6. *Consider the 2-dimensional statistical manifold $(M_2, g, \nabla^{(\alpha)})$ with the skewness operator K . If X^h is a conformal vector field with respect to $(TM_2, g^h, \nabla^{(\alpha)h})$, then X^h reduces to a homothetic vector field. Moreover, X^h is a homothetic vector field if and only if $X^1 = A\theta_1$ and $X^2 = A\theta_2$, where X^1, X^2 are components of X .*

Proof. Let X^h be the horizontal lift of a vector field $X = X^1\partial_1 + X^2\partial_2$, where $\partial_1 = \frac{\partial}{\partial\theta_1}$ and $\partial_2 = \frac{\partial}{\partial\theta_2}$. Considering $i, j = 1, 2$ in (5.4) and using (4.25) and (4.27) we get

$$\frac{X^1}{\theta_1} - \frac{1}{2}(\partial_1 X^1) + \frac{1-\alpha}{2\theta_1}X^1 = -\tilde{\rho}, \quad \frac{X^2}{\theta_2} - \frac{1}{2}(\partial_2 X^2) + \frac{1-\alpha}{2\theta_2}X^2 = -\tilde{\rho}. \tag{5.26}$$

Setting $i = j = m = 1, 2$ in (5.5) implies that $\partial_1(X^1) = \frac{1}{\theta_1}X^1, \partial_2(X^2) = \frac{1}{\theta_2}X^2$ and $\partial_1(X^2) = \partial_2(X^1) = 0$. From these equations we conclude that $X^1 = A\theta_1$ and $X^2 = B\theta_2$. But from (5.26) we derive that $A = B$ and $\tilde{\rho} = (\frac{\alpha}{2} - 1)A$. \square

Let (M, g, ∇) be a statistical manifold with the skewness operator K . Using the splitting (2.4), we can define the Riemannian metric g^S on TM :

$$g^S(X^h, Y^h) = g(X, Y), \quad g^S(X^h, Y^v) = 0, \quad g^S(X^v, Y^v) = g(X, Y), \quad (5.27)$$

which is called the *diagonal lift* or *Sasaki lift* of g [29]. In [23], Matsuzoe and Inoguchi proved that if (M, g, ∇) is a statistical manifold, then (TM, g^S, K^h) is a statistical manifold, where K^h is introduced by (5.2).

Theorem 5.7. *Let (M, g, ∇) be a statistical manifold with the skewness operator K and let X^h be the horizontal lift of a vector field on M . If X^h is conformal with respect to (TM, g^S, K^h) , then X^h is an inessential vector field. Moreover, X^h is an inessential vector field if and only if (5.5), (5.6), (5.7) and the following equations hold:*

$$X^r \nabla_r g_{ij} = 2\tilde{\rho}g_{ij}, \quad (5.28)$$

$$X^r R_{rikj} = 0, \quad (5.29)$$

$$X^r \nabla_r g_{ij} + (\nabla_i X^r)g_{rj} + (\nabla_j X^r)g_{ri} = 2\tilde{\rho}(x)g_{ij}. \quad (5.30)$$

Proof. We can rewrite the metric g^S defined by (5.27) as follows:

$$g^S(\delta_i, \delta_j) = g^S(\partial_{\bar{i}}, \partial_{\bar{j}}) = g_{ij}, \quad g^S(\delta_i, \partial_{\bar{j}}) = g^S(\partial_{\bar{i}}, \delta_j) = 0. \quad (5.31)$$

Using (2.5) and (5.31) we deduce that $L_{X^h}g^S = 2\tilde{\rho}g^S$ if and only if (5.28), (5.29) and (5.30) hold. According to Theorem 5.1, $L_X^h K^h = 0$ if and only if (5.5), (5.6) and (5.7) hold. Also, applying $\partial_{\bar{k}}$ to relation (5.28) implies $\partial_{\bar{k}}\tilde{\rho} = 0$, i.e., $\tilde{\rho}$ depends only on x^h . □

Theorem 5.8. *Consider the generalized Gaussian distribution manifold $(M_1, g, \nabla^{(\alpha)})$. There does not exist any non-zero vector field on M_1 such that X^h is a conformal vector field with respect to (TM_1, g^S, K^h) .*

Proof. Setting $i = j = 1$ in (5.28) we obtain

$$X^2 = -\frac{c_{11}}{c_{11} + c_{121}^{(1)} + \frac{1-\alpha}{2}c_{121}}\tilde{\rho}\sigma.$$

Similarly, applying $i = j = 2$ in (5.28) implies

$$X^2 = -\frac{c_{22}}{c_{22} + c_{222}^{(1)} + \frac{1-\alpha}{2}c_{222}}\tilde{\rho}\sigma. \quad (5.32)$$

Since $\beta \neq 1$, the two equations above imply $\alpha = 2$. Setting $i = 1, j = 2$ in (5.28) and considering $\alpha = 2$ we get

$$X^1 \left(c_{11} + c_{121}^{(1)} - \frac{1}{2}c_{121} \right) = 0,$$

which gives us $X^1 = 0$ (since $\beta \neq 1$, the coefficient of X^1 in the above equation is non-zero). Putting $i = j = 1$ in (5.30) gives us $X^2 = -\tilde{\rho}\sigma$. If $X^2 \neq 0$, using this equation and (5.32) we deduce that $\beta = \frac{1}{2}$, which is a contradiction (because β is an even number). So $X^2 = 0$, and consequently $X = 0$. □

Theorem 5.9. *Consider the 2-dimensional statistical manifold $(M_2, g, \nabla^{(\alpha)})$ with the skewness operator K . If X^h is a conformal vector field with respect to (TM_2, g^S, K^h) , then X^h reduces to a Killing vector field. Moreover, X^h is a Killing vector field if and only if $X = A(\theta_1 + \theta_2)$, where A is a constant function.*

Proof. Setting $i = j = 1$ in (5.28) and using (4.25) and (4.26) we get

$$\frac{\alpha - 2}{\theta_1} X^1 = \tilde{\rho}. \tag{5.33}$$

Similarly, applying $i = j = 1$ in (5.28) implies

$$\frac{\alpha - 2}{\theta_2} X^2 = \tilde{\rho}.$$

The two equations above give

$$X^2 = \frac{\theta_2}{\theta_1} X^1. \tag{5.34}$$

Considering $i = j = 1$ in (5.30) gives us

$$\frac{X^1}{\theta_1} - \partial_1(X^1) = -\tilde{\rho}. \tag{5.35}$$

Considering (5.7) for $i = j = m = 1$ implies

$$\partial_1 X^1 = \frac{X^1}{\theta_1}. \tag{5.36}$$

Similarly, we get

$$\partial_2 X^2 = \frac{X^2}{\theta_2}. \tag{5.37}$$

Using (5.35) and (5.36) we get $\tilde{\rho} = 0$. So X^h reduces to a Killing vector field. Considering $\tilde{\rho} = 0$ in (5.33) implies that $\alpha = 2$ or $X^1 = 0$. If $X^1 = 0$, then from (5.34) we get $X^2 = 0$, and consequently $X = 0$. So, we must have $\alpha = 2$. Setting $i = j = 1, m = 2$ in (5.5) we deduce that $\partial_1 X^2 = 0$. This equation together with (5.37) yields $X^2 = A\theta_2$, where A is a constant. In a similar way, considering (5.5) for $i = m = 1, j = 2$ implies $\partial_2 X^1 = 0$. So from (5.36) we get $X^1 = B\theta_1$, where B is a constant. On the other hand, from (5.34) we deduce that $A = B$. Therefore, we get $X = A(\theta_1 + \theta_2)$. □

Let (M, g, ∇) be a statistical manifold with the skewness operator K . Using (2.7) and (2.8), the complete lift metric g^c with respect to ∇ is described by the formulas

$$g^c(X^c, Y^c) = (g(X, Y))^c, \quad g^c(X^c, Y^v) = (g(X, Y))^v, \quad g^c(X^v, Y^v) = 0$$

for all $X, Y \in \chi(M)$. Also, using (2.7) and (2.8), the complete lift of K is defined by

$$K_{X^c}^c Y^c = (K_X Y)^c, \quad K_{X^c}^c Y^v = (K_X Y)^v, \quad K_{X^v}^c Y^v = 0.$$

It is easy to see that g^c and K^c have the following expressions with respect to $\{\partial_i, \partial_{\bar{i}}\}$:

$$g^c(\partial_i, \partial_j) = y^r \partial_r g_{ij}, \quad g^c(\partial_i, \partial_{\bar{j}}) = g_{ij}, \quad g^c(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0, \tag{5.38}$$

$$K^c_{\partial_i} \partial_j = K^k_{ij} \partial_k + (y^r \partial_r K^k_{ij}) \partial_{\bar{k}}, \quad K^c_{\partial_i} \partial_{\bar{j}} = K^k_{ij} \partial_{\bar{k}}, \quad K^c_{\partial_{\bar{i}}} \partial_{\bar{j}} = 0. \tag{5.39}$$

Remark 5.10. It is easy to see that g^c has the following expression with respect to $\{\delta_i, \delta_{\bar{i}}\}$:

$$g^c(\delta_i, \delta_j) = y^r \nabla_r g_{ij}, \quad g^c(\partial_i, \partial_{\bar{j}}) = g_{ij}, \quad g^c(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0.$$

If ∇ is a statistical connection which is not the Levi-Civita connection, then g^c is different from g^h . But if ∇ is the Levi-Civita connection, then we have $\nabla_r g_{ij} = 0$ and so g^c coincides with g^h .

Lemma 5.11 ([23]). *Let (M, g, ∇) be a statistical manifold with the skewness K . Then (TM, g^c, K^c) is a statistical manifold.*

Here we study $L_{X^c} g^c$ and $L_{X^c} K^c$. Using (4.1) and (5.38) we get

$$\begin{aligned} (L_{X^c} g^c)(\partial_i, \partial_j) &= y^r \{ (\partial_r X^k) \partial_k g_{ij} + X^k \partial_r \partial_k g_{ij} + g_{ik} \partial_r \partial_j X^k \\ &\quad + \partial_j X^k \partial_r g_{ik} + g_{jk} \partial_r \partial_i X^k + \partial_i X^k \partial_r g_{jk} \} \\ &= (L_X g)^c(\partial_i, \partial_j), \end{aligned} \tag{5.40}$$

$$(L_{X^c} g^c)(\partial_i, \partial_{\bar{j}}) = X^k \partial_k g_{ij} + g_{kj} \partial_i X^k + g_{ki} \partial_j X^k = (L_X g)^c(\partial_i, \partial_{\bar{j}}), \tag{5.41}$$

$$(L_{X^c} g^c)(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0 = (L_X g)^c(\partial_{\bar{i}}, \partial_{\bar{j}}). \tag{5.42}$$

Also, using (4.2) and (5.39) we obtain

$$\begin{aligned} (L_{X^c} K^c)(\partial_i, \partial_j) &= y^r \{ X^k \partial_k \partial_r K^m_{ij} - K^k_{ij} \partial_k \partial_r X^m + \partial_r X^k \partial_k K^m_{ij} \\ &\quad - \partial_k X^m \partial_r K^k_{ij} + \partial_i X^k \partial_r K^m_{kj} - K^m_{kj} \partial_i \partial_r X^k \\ &\quad + \partial_j X^k \partial_r K^m_{ki} - K^m_{ki} \partial_j \partial_r X^k \} \partial_{\bar{m}} \\ &= (L_X K)^c(\partial_i, \partial_j), \end{aligned} \tag{5.43}$$

$$\begin{aligned} (L_{X^c} K^c)(\partial_i, \partial_{\bar{j}}) &= \{ X^k \partial_k K^m_{ij} - \partial_k X^m K^k_{ij} + \partial_i X^k K^m_{kj} + \partial_j X^k K^m_{ik} \} \partial_{\bar{m}} \\ &= (L_X K)^c(\partial_i, \partial_{\bar{j}}), \end{aligned} \tag{5.44}$$

$$(L_{X^c} K^c)(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0 = (L_X K)^c(\partial_{\bar{i}}, \partial_{\bar{j}}). \tag{5.45}$$

From (5.40)–(5.45) we conclude the following:

Lemma 5.12. *Let (M, g, K) be a statistical manifold. If g^c and K^c are respectively the complete lifts of g and K , then we have*

$$L_{X^c} g^c = (L_X g)^c, \quad L_{X^c} K^c = (L_X K)^c.$$

Now let X^c be a conformal vector field on (TM, g^c, K^c) . Then there exists a function $\tilde{\rho}$ on TM such that $L_{X^c} g^c = 2\tilde{\rho}g^c$ and $L_{X^c} K^c = 0$. From (5.41) we get

$$X^k \partial_k g_{ij} + g_{kj} \partial_i X^k + g_{ki} \partial_j X^k = 2\tilde{\rho}(x, y) g_{ij}.$$

Differentiating the above equation with respect to y^r implies $\partial_{\bar{r}} \tilde{\rho}(x, y) g_{ij} = 0$. So $\partial_{\bar{r}} \tilde{\rho}(x, y) = 0$, i.e., $\tilde{\rho}$ is a function only with respect to (x) . According to this fact

and relations (5.40)–(5.45) we conclude that X^c is a conformal vector field if and only if

$$\begin{aligned}
 (\partial_r X^k) \partial_k g_{ij} + X^k \partial_r \partial_k g_{ij} + g_{ik} \partial_r \partial_j X^k + \partial_j X^k \partial_r g_{ik} \\
 + g_{jk} \partial_r \partial_i X^k + \partial_i X^k \partial_r g_{jk} = 2\tilde{\rho}(x) \partial_r g_{ij},
 \end{aligned}
 \tag{5.46}$$

$$X^k \partial_k g_{ij} + g_{kj} \partial_i X^k + g_{ki} \partial_j X^k = 2\tilde{\rho}(x) g_{ij},
 \tag{5.47}$$

$$X^k \partial_k K_{ij}^m - \partial_k X^m K_{ij}^k + \partial_i X^k K_{kj}^m + \partial_j X^k K_{ik}^m = 0.
 \tag{5.48}$$

Differentiating (5.47) with respect to x^r we get

$$\begin{aligned}
 (\partial_r X^k) \partial_k g_{ij} + X^k \partial_r \partial_k g_{ij} + g_{ik} \partial_r \partial_j X^k + \partial_j X^k \partial_r g_{ik} \\
 + g_{jk} \partial_r \partial_i X^k + \partial_i X^k \partial_r g_{jk} = 2(\partial_r \tilde{\rho}) g_{ij} + 2\tilde{\rho}(x) \partial_r g_{ij}.
 \end{aligned}
 \tag{5.49}$$

The relations (5.46) and (5.49) imply $(\partial_r \tilde{\rho}) g_{ij} = 0$. So $\partial_r \tilde{\rho} = 0$, i.e., $\tilde{\rho}$ is a constant function. Indeed, conformal vector fields reduce to homothetic (Killing) vector fields. Also, (5.46)–(5.48) reduce to the following:

$$\begin{aligned}
 X^k \partial_k g_{ij} + g_{kj} \partial_i X^k + g_{ki} \partial_j X^k &= 2c g_{ij}, \\
 X^k \partial_k K_{ij}^m - \partial_k X^m K_{ij}^k + \partial_i X^k K_{kj}^m + \partial_j X^k K_{ik}^m &= 0,
 \end{aligned}$$

where c is a constant. But these last two conditions are equivalent to the homothetic (Killing) property of X . Thus we conclude the following:

Theorem 5.13. *Let (M, g, ∇) be a statistical manifold. There does not exist any non-homothetic (non-Killing) conformal complete vector field on (TM, g^c, K^c) . Moreover, X^c is a homothetic (Killing) vector field on (TM, g^c, K^c) if and only if X is a homothetic (Killing) vector field on (M, g, ∇) .*

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Received: August 1, 2020

Accepted: March 9, 2021