REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 65, No. 1, 2023, Pages 67–77
Published online: March 27, 2023
https://doi.org/10.33044/revuma.2620

THE AFFINE LOG-ALEKSANDROV–FENCHEL INEQUALITY

CHANG-JIAN ZHAO

ABSTRACT. We establish a new affine log-Aleksandrov–Fenchel inequality for mixed affine quermassintegrals by introducing new concepts of affine and multiple affine measures, and using the newly established Aleksandrov–Fenchel inequality for multiple mixed affine quermassintegrals. Our new inequality yields as special cases the classical Aleksandrov–Fenchel inequality and the L_p-affine log-Aleksandrov–Fenchel inequality. The affine log-Minkowski and log-Aleksandrov–Fenchel inequalities are also derived.

1. Introduction

In 2016, Stancu [16] established the following logarithmic Minkowski inequality:

The log-Minkowski inequality. *If K and L are convex bodies in $\mathbb{R}^n$ containing the origin in their interior, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\bar{v}_1 \geq \frac{1}{n} \ln\left(\frac{V(K)}{V(L)}\right), \tag{1.1}$$

with equality if and only if K and L are homothetic, where dv_1 is the mixed volume measure, $dv_1 = \frac{1}{n} h_K dS(L,u)$, and $d\bar{v}_1 = \frac{1}{V_1(L,K)} dv_1$ is its normalization, and $V_1(L,K)$ denotes the usual mixed volume of L and K, defined by (see [2])

$$V_1(L,K) = \frac{1}{n}\int_{S^{n-1}} h_K \, dS(L,u).$$

The functions h_K and h_L are support functions of the convex bodies K and L, respectively, and $dS(L,u)$ is the surface area measure of L. If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then (see [15])

$$h_K(x) = \max\{x \cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$, defines the support function $h_K(x)$ of K, and defines the support function $h_K(u)$ on the sphere by restriction to unit vectors u and it is simply written as h_K.

2020 *Mathematics Subject Classification.* 46E30, 52A39, 52A40.

Key words and phrases. Affine quermassintegral, mixed affine quermassintegral, multiple mixed affine quermassintegral, log-Minkowski inequality, log-Aleksandrov–Fenchel inequality.

Research is supported by the National Natural Science Foundation of China (11371334, 10971205).

Associated with the convex bodies $K_1, \dots, K_{n-1}$ in $\mathbb{R}^n$, there exists a unique positive Borel measure on S^{n-1}, $S(K_1, \dots, K_{n-1}; \cdot)$, called the *mixed area measure* of $K_1, \dots, K_{n-1}$, with the property that for any convex body K_n one has the integral representation for the mixed volume (see e.g. [2, p. 354]):

$$V(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} h_{K_n}\, dS(K_1, \dots, K_{n-1}; u).$$

The integration is with respect to the mixed area measure $S(K_1, \dots, K_{n-1}; \cdot)$ on S^{n-1}. The mixed area measure $S(K_1, \dots, K_{n-1}; \cdot)$ is symmetric in its (first $n-1$) arguments. The log-Minkowski inequality is a special case of the following log-Aleksandrov–Fenchel inequality established by Zhao [20].

The log-Aleksandrov–Fenchel inequality. *If $L_1, \dots, L_n, K_n$ are convex bodies in $\mathbb{R}^n$ containing the origin in their interior and $1 \le r \le n$, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_{L_n}}{h_{K_n}}\right) d\overline{V}(L_1, \dots, L_n) \ge \ln\left(\frac{\prod_{i=1}^{r} V(L_i, \dots, L_i, L_{r+1}, \dots, L_n)^{1/r}}{V(L_1, \dots, L_{n-1}, K_n)}\right), \tag{1.2}$$

where $d\overline{V}(L_1, \dots, L_n)$ denotes the multiple mixed volume probability measure of the convex bodies $L_1, \dots, L_n$, defined by

$$d\overline{V}(L_1, \dots, L_n) = \frac{1}{nV(L_1, \dots, L_n)} h(L_n, u)\, dS(L_1, \dots, L_{n-1}; u).$$

Lutwak [9] proposed to define the *affine quermassintegrals* for a convex body K, $\Phi_0(K), \Phi_1(K), \dots, \Phi_n(K)$, by taking $\Phi_0(K) := V(K)$, $\Phi_n(K) := \omega_n$ and, for $0 < j < n$,

$$\Phi_{n-j}(K) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j(K|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi)\right]^{-1/n},$$

where $G_{n,j}$ denotes the Grassmann manifold of j-dimensional subspaces in $\mathbb{R}^n$, μ_j denotes the gauge Haar measure on $G_{n,j}$, $\mathrm{vol}_j(K|\xi)$ denotes the j-dimensional volume of the positive projection of K on j-dimensional subspace $\xi \subset \mathbb{R}^n$ and ω_j denotes the volume of a j-dimensional ball. The *mixed affine quermassintegrals* of j convex bodies $K_1, \dots, K_j$, denoted by $\Phi_{n-j}(K_1 \dots K_j)$, is defined by (see [24])

$$\Phi_{n-j}(K_1 \dots K_j) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j(K_1 \dots K_j|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi)\right]^{-1/n}$$

for $0 < j < n$, where $\mathrm{vol}_j(K_1 \dots K_j|\xi)$ stands for $\mathrm{vol}_j(K_1|\xi, \dots, K_j|\xi)$, the j-dimensional mixed volume of $K_1|\xi, \dots, K_j|\xi$, and by letting

$$\Phi_n(K_1 \dots K_j) := \omega_n$$

and

$$\Phi_0(K_1 \dots K_j) := V(K_1, \dots, K_n).$$

Recently, the log-Minkowski inequality and the log-Aleksandrov–Fenchel inequality and its dual form have attracted extensive attention and research: see references [1, 3, 4, 6, 7, 8, 12, 13, 14, 18, 17, 19, 20, 21, 22, 23]. In this paper, we generalize the log-Minkowski inequality (1.1) and the log-Aleksandrov–Fenchel inequality (1.2) to the mixed affine quermassintegrals. The following affine log-Aleksandrov–Fenchel inequality is established by introducing the concepts of affine and multiple mixed affine measures and using the newly established Aleksandrov–Fenchel inequality for multiple mixed affine quermassintegrals.

Theorem 1.1 (The affine log-Aleksandrov–Fenchel inequality). *If $K_1, \dots, K_j$ are convex bodies containing the origin, L_j is a convex body containing the origin in its interior, $0 < j \le n$, and $0 < r \le j$, then*

$$\int_{G_{n,j}} \ln\left(\frac{\mathrm{vol}_j(K_1 \dots K_j|\xi)}{\mathrm{vol}_j(K_1 \dots K_{j-1}L_j|\xi)}\right) d\overline{\Phi}^{-n}_{n-j}(K_1 \dots K_j L_j) \ge \ln\left(\frac{\prod_{i=1}^{r} \Phi_{n-j}(K_i \dots K_i K_{r+1} \cdots K_j)^{1/r}}{\Phi_{n-j}(K_1 \dots K_{j-1}L_j)}\right), \quad (1.3)$$

where $d\overline{\Phi}_{n-j}(K_1 \dots K_j L_j)$ denotes a new multiple mixed affine probability measure of $K_1, \dots, K_j, L_j$, defined by (see Section 3)

$$d\overline{\Phi}^{-n}_{n-j}(K_1 \dots K_j L_j) = \frac{1}{\overline{\Phi}^{-n}_{n-j}(K_1 \dots K_j L_j)} d\overline{\varphi}^{-n}_{n-j}(K_1 \dots K_j L_j), \quad (1.4)$$

where $d\overline{\varphi}_{n-j}(K_1 \dots K_j L_j)$ denotes the multiple mixed affine measure, defined by

$$d\overline{\varphi}^{-n}_{n-j}(K_1 \dots K_j L_j) = \frac{\mathrm{vol}_j(K_1 \dots K_j|\xi)}{\mathrm{vol}_j(K_1 \dots K_{j-1}L_j|\xi)} \left(\frac{\omega_n \mathrm{vol}_j(K_1 \dots K_{j-1}L_j|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi),$$

and $\overline{\Phi}_{n-j}(K_1 \dots K_j L_j)$ is the multiple affine quermassintegral of $K_1, \dots, K_j, L_j$, defined by (see [24]*)*

$$\overline{\Phi}^{-n}_{n-j}(K_1 \dots K_j L_j) = \int_{G_{n,j}} \frac{\mathrm{vol}_j(K_1 \dots K_j|\xi)}{\mathrm{vol}_j(K_1 \dots K_{j-1}L_j|\xi)} \left(\frac{\omega_n \mathrm{vol}_j(K_1 \dots K_{j-1}L_j|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi).$$

When $j = n$ and $K_j = L_j$, (1.3) becomes the following classical Aleksandrov–Fenchel inequality for convex bodies.

Corollary 1.2 (The Aleksandrov–Fenchel inequality). *If $K_1, \dots, K_n$ are convex bodies in $\mathbb{R}^n$ containing the origin, $1 < i \le r$, and $0 < r \le n$, then*

$$V(K_1, \dots, K_n) \ge \prod_{i=1}^{r} V(K_i, \dots, K_i, K_{r+1}, \dots, K_n)^{1/r}$$

(see e.g. [15]*).*

A special case of (1.3) is the following affine log-Minkowski inequality for affine quermassintegrals.

Corollary 1.3 (The affine log-Minkowski inequality)**.** *If K is a convex body in $\mathbb{R}^n$ containing the origin, L is a convex body in $\mathbb{R}^n$ containing the origin in its interior, and $0 < j \leq n$, then*

$$\int_{G_{n,j}} \ln\left(\frac{\mathrm{vol}_j(L\ldots LK|\xi)}{\mathrm{vol}_j(L|\xi)}\right) d\Phi^{-n}_{n-j}(L,K) \geq \frac{1}{j}\ln\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(L)}\right),$$

with equality if and only if L and K are homothetic. Here $\mathrm{vol}_j(L\ldots LK|\xi)$ denotes $\mathrm{vol}_j(\underbrace{L\ldots L}_{j-1}K|\xi)$; $d\Phi_{n-j}(L,K)$ denotes a new affine probability measure of convex bodies L and K, defined by

$$d\Phi^{-n}_{n-j}(L,K) = \frac{1}{\Phi^{-n}_{n-j}(L,K)} d\varphi^{-n}_{n-j}(L,K),$$

where $d\varphi_{n-j}(L,K)$ denotes the affine measure, defined by

$$d\varphi^{-n}_{n-j}(L,K) = \frac{\mathrm{vol}_j(L\ldots LK|\xi)}{\mathrm{vol}_j(L|\xi)}\left(\frac{\omega_n \mathrm{vol}_j(L|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi);$$

and $\Phi_{n-j}(L,K)$ is the mixed affine quermassintegral of L and K, defined by

$$\Phi_{n-j}(L,K) = \omega_n\left[\int_{G_{n,j}} \frac{\mathrm{vol}_j(L\ldots LK|\xi)}{\mathrm{vol}_j(L|\xi)}\left(\frac{\mathrm{vol}_j(L|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi)\right]^{-1/n}.$$

Obviously, (1.2) is also a special case of (1.3).

2. Notations and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. A *body* in $\mathbb{R}^n$ is a compact set with the usual open set topology and a *convex body* in $\mathbb{R}^n$ is a compact convex set with non-empty interior. Let $\mathcal{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$, let $\mathcal{K}^n_o$ be the class of members of $\mathcal{K}^n$ containing the origin, and let $\mathcal{K}^n_{oo}$ be those sets in $\mathcal{K}^n$ containing the origin in their interiors. We reserve the letter $u \in S^{n-1}$ for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. For a compact set K, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the *volume* of K. Let d denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$,

$$d(K,L) = |h_K - h_L|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$. Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. If ξ is a subspace of $\mathbb{R}^n$, then it is easy to show that

$$h_{K|\xi} = h_K.$$

Let $\varphi : [0,\infty) \to (0,\infty)$ be a convex and increasing function such that $\varphi(1) = 1$ and $\varphi(0) = 0$. Let Φ denote the set of convex functions $\varphi : [0,\infty) \to [0,\infty)$ that are increasing and satisfy $\varphi(0) = 0$ and $\varphi(1) = 1$.

2.1. **Mixed volumes.** If, for $i = 1, 2, \ldots, r$, $K_i \in \mathcal{K}^n$ and λ_i is a nonnegative real number, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in λ_i given by (see e.g. [10])

$$V(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1,\ldots,i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{2.1}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$ and is uniquely determined by (2.1); it is called the *mixed volume* of $K_i, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$; then the mixed volume $V(K_1, \ldots, K_n)$ is written as $V_i(K, L)$. If $K_1 = \cdots = K_{n-i} = K$, then $K_{n-i+1} = \cdots = K_n = B$. The mixed volume $V_i(K, B)$ is written as $W_i(K)$ and called *quermassintegrals* (or *ith mixed quermassintegrals*) of K. We write $W_i(K, L)$ for the mixed volume $V(K, \ldots, K, \underbrace{B, \ldots, B}_{i}, L)$, called *mixed quermassintegrals*, and

$$W_i(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L \, dS_i(K, u).$$

Associated with $K_1, \ldots, K_n \in \mathcal{K}^n$ is a Borel measure $S(K_1, \ldots, K_{n-1}, \cdot)$ on S^{n-1}, called the *mixed surface area measure* of $K_1, \ldots, K_{n-1}$, which has the property that for each $K \in \mathcal{K}^n$,

$$V(K_1, \ldots, K_{n-1}, K) = \frac{1}{n} \int_{S^{n-1}} h_K \, dS(K_1, \ldots, K_{n-1}, u).$$

Let $K_1 = \cdots = K_{n-i-1} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$; then the mixed surface area measure $S(K_1, \ldots, K_{n-1}, \cdot)$ is written as $S(K[n-i], L[i], \cdot)$. When $L = B$, $S(K[n-i], L[i], \cdot)$ is written as $S_i(K, \cdot)$ and called *ith mixed surface area measure.*

2.2. **The multiple mixed affine quermassintegrals.** In [24], as a special case of Orlicz multiple mixed affine quermassintegrals, the multiple mixed affine quermassintegrals was introduced as follows:

Definition 2.1 (The multiple mixed affine quermassintegrals)**.** If $0 \leq j \leq n$, $K_1, \ldots, K_j \in \mathcal{K}^n_o$, and $L_j \in \mathcal{K}^n_{oo}$, the *multiple mixed affine quermassintegral* of $K_1, \ldots, K_j, L_j$, denoted by $\overline{\Phi}_{n-j}(K_1 \ldots K_j L_j)$, is defined by

$$\begin{aligned}
&\overline{\Phi}_{n-j}(K_1 \ldots K_j L_j) \\
&= \omega_n \left[\int_{G_{n,j}} \frac{\mathrm{vol}_j(K_1 \ldots K_j|\xi)}{\mathrm{vol}_j(K_1 \ldots K_{j-1} L_j|\xi)} \left(\frac{\mathrm{vol}_j(K_1 \ldots K_{j-1} L_j|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n}.
\end{aligned}$$

When $K_j = L_j$, $\overline{\Phi}_{n-j}(K_1 \ldots K_j L_j)$ becomes the mixed affine quermassintegral $\Phi_{n-j}(K_1 \ldots K_j)$. When $K_1 = \cdots = K_j = L_j = K$, $\overline{\Phi}_{n-j}(K_1 \ldots K_j L_j)$ becomes the well-known affine quermassintegral $\Phi_{n-j}(K)$ of K. When $K_1 = \cdots = K_j =$

K and $L_j = L$, $\overline{\Phi}_{n-j}(K_1 \ldots K_j L_j)$ becomes the mixed affine quermassintegral $\Phi_{n-j}(K, L)$ of K and L. Specifically, for $j = n$, we define

$$\overline{\Phi}_0(K_1 \ldots K_n L_n) = \left(\frac{V(K_1, \ldots, K_n)}{V(K_1, \ldots, K_{n-1}, L_n)} \right)^{-1/n} V(K_1, \ldots, K_{n-1}, L_n).$$

In [24], Zhao proved also that the multiple affine quermassintegrals is a first order variation of the mixed affine quermassintegral of j convex bodies. For $K_1, \ldots, K_j \in \mathcal{K}_o^n$, $L_j \in \mathcal{K}_{oo}^n$, $0 \le j \le n$, and $\varepsilon > 0$,

$$\begin{aligned}\overline{\Phi}_{n-j}(K_1 \ldots K_j L_j)^{-n} &= \Phi_{n-j}(K_1 \ldots K_{j-1} L_j)^{-(1+n)} \\ &\quad \times \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \Phi_{n-j}(K_1 \ldots K_{j-1}(L_j + \varepsilon \cdot K_j)).\end{aligned}$$

3. The affine log-Aleksandrov–Fenchel inequality

In this section, in order to derive the affine log-Aleksandrov–Fenchel inequality, we need to define some new mixed affine measures. From the definition of mixed affine quermassintegrals, we introduce the following mixed affine measure of convex bodies $K_1, \ldots, K_j$.

Definition 3.1 (Mixed affine measure)**.** For $K_1, \ldots, K_j \in \mathcal{K}_o^n$ and $0 < j \le n$, the *mixed affine measure* of $K_1, \ldots, K_j$, denoted by $d\varphi_{n-j}(K_1 \ldots K_j)$, is defined by

$$d\varphi_{n-j}^{-n}(K_1 \ldots K_j) = \left(\frac{\omega_n \mathrm{vol}_j(K_1 \ldots K_j|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi). \tag{3.1}$$

From Definition 3.1, we find the following mixed affine probability measure:

$$d\Phi_{n-j}^{-n}(K_1 \ldots K_j) = \frac{1}{\Phi_{n-j}^{-n}(K_1 \ldots K_j)} d\varphi_{n-j}^{-n}(K_1 \ldots K_j).$$

From the definition of multiple mixed affine quermassintegrals, we introduce the following multiple mixed affine measure of convex bodies.

Definition 3.2 (Multiple mixed affine measure)**.** For $K_1, \ldots, K_j \in \mathcal{K}_o^n$, $L_j \in \mathcal{K}_{oo}^n$, and $0 \le j \le n$, the *multiple affine measure* of $K_1, \ldots, K_j, L_j$, denoted by $d\overline{\varphi}_{n-j}(K_1 \ldots K_j L_j)$, is defined by

$$\begin{aligned}&d\overline{\varphi}_{n-j}^{-n}(K_1 \ldots K_j L_j) \\ &\quad := \frac{\mathrm{vol}_j(K_1 \ldots K_j|\xi)}{\mathrm{vol}_j(K_1 \ldots K_{j-1} L_j|\xi)} \left(\frac{\omega_n \mathrm{vol}_j(K_1 \ldots K_{j-1} L_j|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi).\end{aligned} \tag{3.2}$$

From Definition 3.2, the *multiple mixed affine probability measure* is defined by

$$d\overline{\Phi}_{n-j}^{-n}(K_1 \ldots K_j L_j) = \frac{1}{\overline{\Phi}_{n-j}^{-n}(K_1 \ldots K_j L_j)} d\overline{\varphi}_{n-j}^{-n}(K_1 \ldots K_j L_j). \tag{3.3}$$

Lemma 3.3 (The Aleksandrov–Fenchel inequality for multiple mixed affine quermassintegrals [24]). *If $K_1,\dots,K_j \in \mathcal{K}_o^n$, $L_j \in \mathcal{K}_{oo}^n$, $0 \le j \le n$, and $0 < r \le j$, then*

$$\left(\frac{\overline{\Phi}_{n-j}(K_1,\dots,K_j,L_j)}{\Phi_{n-j}(K_1,\dots,K_{j-1},L_j)}\right)^{-n} \ge \frac{\prod_{i=1}^r \Phi_{n-j}(K_i,\dots,K_i,K_{r+1},\dots,K_j)^{1/r}}{\Phi_{n-j}(K_1,\dots,K_{j-1},L_j)}.$$

Theorem 3.4 (The affine log-Aleksandrov–Fenchel inequality). *If $K_1\dots K_j \in \mathcal{K}_o^n$, $L_j \in \mathcal{K}_{oo}^n$, $0 < j \le n$, and $0 < r \le j$, then*

$$\int_{G_{n,j}} \ln\left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right) d\overline{\Phi}_{n-j}^{-n}(K_1\dots K_jL_j) \ge \ln\left(\frac{\prod_{i=1}^r \Phi_{n-j}(K_i\dots K_iK_{r+1}\cdots K_j)^{1/r}}{\Phi_{n-j}(K_1\dots K_{j-1}L_j)}\right),$$

where $d\overline{\Phi}_{n-j}(K_1\dots K_jL_j)$ is as in (1.4).

Proof. From (3.1), (3.2), and (3.3), we obtain

$$\int_{G_{n,j}} \frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)} \ln\left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right) d\varphi_{n-j}^{-n}(K_1\dots K_{j-1}L_j) = \int_{G_{n,j}} \ln\left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right) d\overline{\varphi}_{n-j}^{-n}(K_1\dots K_jL_j).$$

Note the following equality:

$$\overline{\Phi}_{n-j}^{-n}(K_1\dots K_jL_j) = \int_{G_{n,j}} \frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)} \left(\frac{\omega_n \mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi).$$

From Lebesgue's dominated convergence theorem, we obtain

$$\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right)^{\frac{q}{q+n}} d\varphi_{n-j}^{-n}(K_1\dots K_{j-1}L_j) \to \overline{\Phi}_{n-j}^{-n}(K_1\dots K_jL_j)$$

as $q \to \infty$, and

$$\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right)^{\frac{q}{q+n}} \ln\left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right) d\varphi_{n-j}^{-n}(K_1\dots K_{j-1}L_j) \to \int_{G_{n,j}} \ln\left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right) d\overline{\varphi}_{n-j}^{-n}(K_1\dots K_jL_j)$$

as $q \to \infty$.

On the other hand, define the function $g_{L,K}(q) : [1,\infty] \to \mathbb{R}$ by

$$g_{L,K}(q) = \frac{1}{\overline{\Phi}_{n-j}^{-n}(K_1\dots K_jL_j)} \times \int_{G_{n,j}} \left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right)^{\frac{q}{q+n}} d\varphi_{n-j}^{-n}(K_1\dots K_{j-1}L_j). \tag{3.4}$$

From (3.4), we obtain

$$\begin{aligned}\frac{dg_{L,K}(q)}{dq} = \frac{n}{(q+n)^2\overline{\Phi}_{n-j}^{-n}(K_1\ldots K_jL_j)}\int_{G_{n,j}}\left(\frac{\mathrm{vol}_j(K_1\ldots K_j|\xi)}{\mathrm{vol}_j(K_1\ldots K_{j-1}L_j|\xi)}\right)^{\frac{q}{q+n}}\\ \times\ln\left(\frac{\mathrm{vol}_j(K_1\ldots K_j|\xi)}{\mathrm{vol}_j(K_1\ldots K_{j-1}L_j|\xi)}\right)d\varphi_{n-j}^{-n}(K_1\ldots K_{j-1}L_j).\end{aligned}\tag{3.5}$$

and

$$\lim_{q\to\infty} g_{L,K}(q) = 1.\tag{3.6}$$

From (3.4), (3.5), and (3.6), we have

$$\begin{aligned}&\lim_{q\to\infty}\ln(g_{L,K}(q))^{q+n} = -(q+n)^2\lim_{q\to\infty}\frac{1}{g_{L,K}(q)}\frac{dg_{L,K}(q)}{dq}\\ &\quad= -\frac{n}{\overline{\Phi}_{n-j}^{-n}(K_1\ldots K_jL_j)}\\ &\quad\times\lim_{q\to\infty}\frac{\int_{G_{n,j}}\left(\frac{\mathrm{vol}_j(K_1\ldots K_j|\xi)}{\mathrm{vol}_j(K_1\ldots K_{j-1}L_j|\xi)}\right)^{\frac{q}{q+n}}\ln\left(\frac{\mathrm{vol}_j(K_1\ldots K_j|\xi)}{\mathrm{vol}_j(K_1\ldots K_{j-1}L_j|\xi)}\right)d\varphi_{n-j}^{-n}(K_1\ldots K_{j-1}L_j)}{g_{L,K}(q)}\\ &\quad= -\frac{n}{\overline{\Phi}_{n-j}^{-n}(K_1\ldots K_jL_j)}\\ &\quad\times\int_{G_{n,j}}\frac{\mathrm{vol}_j(K_1\ldots K_j|\xi)}{\mathrm{vol}_j(K_1\ldots K_{j-1}L_j|\xi)}\ln\left(\frac{\mathrm{vol}_j(K_1\ldots K_j|\xi)}{\mathrm{vol}_j(K_1\ldots K_{j-1}L_j|\xi)}\right)d\varphi_{n-j}^{-n}(K_1\ldots K_{j-1}L_j).\end{aligned}$$

Hence

$$\begin{aligned}&\exp\Bigg(-\frac{n}{\overline{\Phi}_{n-j}^{-n}(K_1\ldots K_jL_j)}\int_{G_{n,j}}\frac{\mathrm{vol}_j(K_1\ldots K_j|\xi)}{\mathrm{vol}_j(K_1\ldots K_{j-1}L_j|\xi)}\\ &\qquad\times\ln\left(\frac{\mathrm{vol}_j(K_1\ldots K_j|\xi)}{\mathrm{vol}_j(K_1\ldots K_{j-1}L_j|\xi)}\right)d\varphi_{n-j}^{-n}(K_1\ldots K_{j-1}L_j)\Bigg)\\ &\quad=\lim_{q\to\infty}(g_{L,K})^{q+n}\\ &\quad=\lim_{q\to\infty}\Bigg(\frac{1}{\overline{\Phi}_{n-j}^{-n}(K_1\ldots K_jL_j)}\\ &\qquad\times\int_{G_{n,j}}\left(\frac{\mathrm{vol}_j(K_1\ldots K_j|\xi)}{\mathrm{vol}_j(K_1\ldots K_{j-1}L_j|\xi)}\right)^{\frac{q}{q+n}}d\varphi_{n-j}^{-n}(K_1\ldots K_{j-1}L_j)\Bigg)^{q+n}.\end{aligned}$$

From Hölder's inequality, we obtain

$$\left(\int_{G_{n,j}}\left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right)^{\frac{q}{q+n}} d\varphi^{-n}_{n-j}(K_1\dots K_{j-1}L_j)\right)^{(q+n)/q}$$

$$\times\left(\int_{G_{n,j}} d\varphi^{-n}_{n-j}(K_1\dots K_{j-1}L_j)\right)^{-n/q}$$

$$\le\int_{G_{n,j}}\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}d\varphi^{-n}_{n-j}(K_1\dots K_{j-1}L_j)$$

$$=\overline{\Phi}^{-n}_{n-j}(K_1\dots K_jL_j).$$

Hence

$$\left(\frac{1}{\overline{\Phi}^{-n}_{n-j}(K_1\dots K_jL_j)}\int_{G_{n,j}}\left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right)^{\frac{q}{q+n}} d\varphi^{-n}_{n-j}(K_1\dots K_{j-1}L_j)\right)^{q+n}$$

$$\le\left(\frac{\Phi^{n}_{n-j}(K_1\dots K_{j-1}L_j)}{\overline{\Phi}^{-n}_{n-j}(K_1\dots K_jL_j)}\right)^{-n}.$$

Therefore

$$\exp\Bigg(-\frac{n}{\overline{\Phi}^{-n}_{n-j}(K_1\dots K_jL_j)}\int_{G_{n,j}}\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}$$

$$\times\ln\left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right)d\varphi^{-n}_{n-j}(K_1\dots K_{j-1}L_j)\Bigg)$$

$$\le\left(\frac{\Phi^{-n}_{n-j}(K_1\dots K_{j-1}L_j)}{\overline{\Phi}^{-n}_{n-j}(K_1\dots K_jL_j)}\right)^{-n}.$$

Hence

$$\frac{1}{\Phi^{-n}_{\varphi,n-j}(L,K)}$$

$$\times\int_{G_{n,j}}\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\ln\left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right)d\varphi^{-n}_{n-j}(K_1\dots K_{j-1}L_j)$$

$$\ge\ln\left(\frac{\overline{\Phi}^{-n}_{n-j}(K_1\dots K_jL_j)}{\Phi^{-n}_{n-j}(K_1\dots K_{j-1}L_j)}\right).$$

That is,

$$\int_{G_{n,j}}\ln\left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right)d\overline{\Phi}^{-n}_{n-j}(K_1\dots K_jL_j)$$

$$\ge\ln\left(\frac{\overline{\Phi}^{-n}_{n-j}(K_1\dots K_jL_j)}{\Phi^{-n}_{n-j}(K_1\dots K_{j-1}L_j)}\right).$$

Further, by using the Aleksandrov–Fenchel inequality for multiple mixed affine quermassintegrals in Lemma 3.3, we obtain

$$\int_{G_{n,j}} \ln\left(\frac{\mathrm{vol}_j(K_1\dots K_j|\xi)}{\mathrm{vol}_j(K_1\dots K_{j-1}L_j|\xi)}\right) d\overline{\Phi}_{n-j}^{-n}(K_1\dots K_j L_j)$$
$$\geq \ln\left(\frac{\prod_{i=1}^{r}\Phi_{n-j}(K_i\dots K_i K_{r+1}\dots K_j)^{1/r}}{\Phi_{n-j}(K_1\dots K_{j-1}L_j)}\right).$$

This completes the proof. □

Acknowledgments

The author expresses his grateful thanks to the referee for his many excellent suggestions and comments.

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Chang-Jian Zhao
Department of Mathematics, China Jiliang University, Hangzhou 310018, P. R. China
chjzhao@163.com, chjzhao@cjlu.edu.cn

Received: April 10, 2021
Accepted: September 21, 2021