REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 65, No. 2, 2023, Pages 587–609
Published online: December 29, 2023
https://doi.org/10.33044/revuma.2812

SOME INEQUALITIES FOR LAGRANGIAN SUBMANIFOLDS IN HOLOMORPHIC STATISTICAL MANIFOLDS OF CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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Abstract. We obtain two types of inequalities for Lagrangian submanifolds in holomorphic statistical manifolds of constant holomorphic sectional curvature. One relates the Oprea invariant to the mean curvature, the other relates the Chen invariant to the mean curvature. Our results generalize the corresponding inequalities for Lagrangian submanifolds in complex space forms.

1. Introduction

In the theory of submanifolds, geometric quantities have been classified into two types: intrinsic invariants and extrinsic invariants. The classical intrinsic invariants include the sectional curvature, the Ricci curvature, and the scalar curvature. B. Y. Chen [4] introduced a new type of intrinsic invariant by combining the scalar curvature and the sectional curvature in a specific way, nowadays called Chen invariant. Following this idea, T. Oprea [16] also introduced a new type of invariant, nowadays called Oprea invariant. On the other hand, the classical extrinsic invariants mainly include the mean curvature, the normal scalar curvature and the Casorati curvature. Establishing inequalities between the intrinsic invariants and the extrinsic invariants has always been a fundamental problem in the theory of submanifolds. In the early time, various inequalities were found for submanifolds in real space forms [5, 8]. For complex space forms, some special types of submanifolds can be explored, such as Lagrangian submanifolds. In [17, 18], T. Oprea established inequalities between the Oprea invariant and the mean curvature for this kind of submanifolds by using the optimization method on Riemannian manifolds.

2020 *Mathematics Subject Classification.* 53C15, 53B25, 53B35.
Key words and phrases. Holomorphic statistical manifold, Lagrangian submanifold, Oprea invariant, Chen invariant.

Theorem 1.1 ([17]). *Let M^n be a Lagrangian submanifold in a complex space form $\tilde{M}^{2n}(4\tilde{c})$, $n \geqslant 3$. Then*

$$\delta_2(M) \leqslant \frac{(n+1)(n-2)}{2}\tilde{c} + \frac{n^2(2n-3)}{2(2n+3)}\|H\|^2, \tag{1.1}$$

where $\delta_2(M)$ is the Oprea invariant of M and H is the mean curvature vector of M.

Theorem 1.2 ([18]). *Let M^n be a Lagrangian submanifold in a complex space form $\tilde{M}^{2n}(4\tilde{c})$, $n \geqslant 3$. Then*

$$\delta_n(M) \leqslant \frac{(n+1)(n-2)}{2}\tilde{c} + \frac{(3n-1)(n-2)n^2}{2(3n+5)(n-1)}\|H\|^2, \tag{1.2}$$

where $\delta_n(M)$ is the Oprea invariant of M and H is the mean curvature vector of M.

In [7], B. Y. Chen and F. Dillen established an inequality between the Chen invariant and the mean curvature for Lagrangian submanifolds in complex space forms.

Theorem 1.3 ([7]). *Let M^n be a Lagrangian submanifold in a complex space form $\tilde{M}^{2n}(4\tilde{c})$. Then*

$$\begin{aligned}\delta(n_1,\ldots,n_k) \leqslant\ & \frac{n^2\left(n-\sum_{i=1}^k n_i + 3k - 1 - 6\sum_{i=1}^k \frac{1}{2+n_i}\right)}{2\left(n-\sum_{i=1}^k n_i + 3k + 2 - 6\sum_{i=1}^k \frac{1}{2+n_i}\right)}\|H\|^2 \\ & + \frac{1}{2}\left[n(n-1) - \sum_{i=1}^k n_i(n_i-1)\right]\tilde{c},\end{aligned} \tag{1.3}$$

where $\delta(n_1,\ldots,n_k)$ is the Chen invariant and H is the mean curvature vector of M.

On the other hand, S. Amari in 1985 [1] introduced the notion of statistical manifolds. From then on, the geometry of statistical manifolds has developed in close relation with affine differential geometry [14] and Hessian geometry [19]. By definition, statistical structures can be considered as a generalization of the Riemannian structures. Recently, the Casorati curvature inequality, the Ricci curvature inequality, and the DDVV inequality have been established for submanifolds in statistical manifolds of constant curvature [13, 2, 3]. In 2009, H. Furuhata introduced the concept of holomorphic statistical manifolds, and focused on holomorphic statistical manifolds of constant holomorphic sectional curvature, which can be viewed as a generalization of complex space forms [9, 10].

The main purpose of this paper is to generalize Theorems 1.1–1.3 to Lagrangian submanifolds in holomorphic statistical manifolds of constant holomorphic sectional curvature. In Section 2, we review some basics of holomorphic statistical manifolds and statistical submanifolds. In Section 3, we establish the inequalities relating Oprea invariants δ_l ($l = 2, \ldots, n$) to the mean curvature (see Theorem 3.1 and Theorem 3.2). We remark that when the ambient space becomes the complex space form and $l = 2$ or $l = n$, our results coincide with Theorem 1.1 and Theorem 1.2, respectively. In Section 4, we establish the inequality relating the Chen invariant $\delta(n_1,\ldots,n_k)$ to the mean curvature (see Theorem 4.1). We remark that when the ambient space becomes the complex space form, our results coincide with Theorem 1.3.

2. Preliminaries

Let $(\tilde{M},\tilde{g})$ be a Riemannian manifold and $\tilde{\nabla}^0$ be the Levi-Civita connection of $\tilde{g}$ on $\tilde{M}$. Throughout this paper, we denote the set of all smooth tangent vector fields on $\tilde{M}$ by $C^\infty(T\tilde{M})$.

We first review some knowledge of statistical manifolds.

Definition 2.1 ([10])**.** Let $\tilde{\nabla}$ be an affine connection on a Riemannian manifold $(\tilde{M},\tilde{g})$. The affine connection $\tilde{\nabla}^*$ is called the *dual connection* of $\tilde{\nabla}$ with respect to $\tilde{g}$ if

$$Z\tilde{g}(X,Y)=\tilde{g}(\tilde{\nabla}_Z X,Y)+\tilde{g}(X,\tilde{\nabla}^*_Z Y) \tag{2.1}$$

for any $X,Y,Z\in C^\infty(T\tilde{M})$.

Obviously, $(\tilde{\nabla}^*)^*=\tilde{\nabla}$. Moreover, if $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are both torsion-free, then [14]

$$\tilde{\nabla}+\tilde{\nabla}^*=2\tilde{\nabla}^0, \tag{2.2}$$

where $\tilde{\nabla}^0$ is the Levi-Civita connection of $\tilde{g}$ on $\tilde{M}$.

Definition 2.2 ([14])**.** Let $(\tilde{M},\tilde{g})$ be a Riemannian manifold and $\tilde{\nabla}$ be an affine connection on $\tilde{M}$. The pair $(\tilde{\nabla},\tilde{g})$ is called a *statistical structure* or a *Codazzi structure* if $\tilde{\nabla}$ is torsion-free and the Codazzi equation

$$(\tilde{\nabla}_X\tilde{g})(Y,Z)=(\tilde{\nabla}_Y\tilde{g})(X,Z)$$

holds for any $X,Y,Z\in C^\infty(T\tilde{M})$. In this case, $(\tilde{M},\tilde{\nabla},\tilde{g})$ is said to be a *statistical manifold* or a *Codazzi manifold*.

By definition, a Riemannian structure $(\tilde{\nabla}^0,\tilde{g})$ is a special statistical structure, which is called a *Riemannian statistical structure* or a *trivial statistical structure* [9]. In fact, the Levi-Civita connection $\tilde{\nabla}^0$ is self-dual with respect to the Riemannian metric $\tilde{g}$. Besides, if $(\tilde{\nabla},\tilde{g})$ is a statistical structure on $\tilde{M}$, so is $(\tilde{\nabla}^*,\tilde{g})$.

Proposition 2.3 ([11])**.** *Let $(\tilde{M},\tilde{\nabla},\tilde{g})$ be a statistical manifold and $\tilde{\nabla}^0$ be the Levi-Civita connection of $\tilde{g}$ on $\tilde{M}$. For any $X,Y,Z\in C^\infty(T\tilde{M})$, the tensor field K of type $(1,2)$ defined by $K:=\tilde{\nabla}-\tilde{\nabla}^0$ satisfies*

$$K_XY=K_YX,\quad \tilde{g}(K_XY,Z)=\tilde{g}(K_XZ,Y). \tag{2.3}$$

Conversely, if a $(1,2)$-tensor field K on $\tilde{M}$ satisfies (2.3), then $(\tilde{M},\tilde{\nabla}^0+K,\tilde{g})$ is a statistical manifold.

Definition 2.4 ([11])**.** Let $(\tilde{M},\tilde{\nabla},\tilde{g})$ be a statistical manifold and $\tilde{\nabla}^*$ be the dual connection of $\tilde{\nabla}$ with respect to $\tilde{g}$. Denote the curvature tensor field of $\tilde{\nabla}$ (resp., $\tilde{\nabla}^*$) by $\tilde{R}$ (resp., $\tilde{R}^*$), i.e., for any $X,Y,Z\in C^\infty(T\tilde{M})$,

$$\tilde{R}(X,Y)Z=\tilde{\nabla}_X\tilde{\nabla}_YZ-\tilde{\nabla}_Y\tilde{\nabla}_XZ-\tilde{\nabla}_{[X,Y]}Z,$$
$$\tilde{R}^*(X,Y)Z=\tilde{\nabla}^*_X\tilde{\nabla}^*_YZ-\tilde{\nabla}^*_Y\tilde{\nabla}^*_XZ-\tilde{\nabla}^*_{[X,Y]}Z.$$

Define

$$\tilde{S}(X,Y)Z=\frac{1}{2}\{\tilde{R}(X,Y)Z+\tilde{R}^*(X,Y)Z\};\tag{2.4}$$

then we call $\tilde{S}$ the *statistical curvature tensor field* of $(\tilde{M},\tilde{\nabla},\tilde{g})$.

Obviously, the statistical curvature tensor field corresponding to the Levi-Civita connection is the *Riemannian curvature tensor field*.

Next, we review some basics of holomorphic statistical manifolds. Firstly, we recall the definition of Kähler manifolds.

Definition 2.5 ([22])**.** Let $(\tilde{M},\tilde{g})$ be an even dimensional Riemannian manifold, $\tilde{\nabla}^0$ be the Levi-Civita connection of $\tilde{g}$, and J be a $(1,1)$-tensor field on $\tilde{M}$. If

$$J^2=-I,\quad \tilde{g}(JX,JY)=\tilde{g}(X,Y),\quad \tilde{\nabla}^0_X JY=J\tilde{\nabla}^0_X Y\tag{2.5}$$

for any $X,Y\in C^\infty(T\tilde{M})$, then $(\tilde{M},J,\tilde{g},\tilde{\nabla}^0)$ is called a *Kähler manifold*.

Definition 2.6 ([22])**.** Let $(\tilde{M},J,\tilde{g},\tilde{\nabla}^0)$ be a Kähler manifold, and $\tilde{R}^0$ be the curvature tensor field of $\tilde{\nabla}^0$. Then $(\tilde{M},J,\tilde{g},\tilde{\nabla}^0)$ is said to be of *constant holomorphic sectional curvature* $\tilde{c}\in\mathbb{R}$ if

$$\begin{aligned}\tilde{R}^0(X,Y)Z=\frac{\tilde{c}}{4}\big\{&\tilde{g}(Y,Z)X-\tilde{g}(X,Z)Y+\tilde{g}(JY,Z)JX\\&-\tilde{g}(JX,Z)JY-2\tilde{g}(JX,Y)JZ\big\}\end{aligned}\tag{2.6}$$

for any $X,Y,Z\in C^\infty(T\tilde{M})$.

A Kähler manifold of constant holomorphic sectional curvature is usually called a *complex space form*.

T. Kurose [12] introduced the notion of a holomorphic statistical manifold by endowing a Kähler manifold with a suitable statistical structure.

Definition 2.7 ([12])**.** Let $(\tilde{M},J,\tilde{g},\tilde{\nabla}^0)$ be a Kähler manifold and $(\tilde{\nabla},\tilde{g})$ be a statistical structure on $\tilde{M}$. Then $(\tilde{M},J,\tilde{g},\tilde{\nabla})$ is called a *holomorphic statistical manifold* if the difference tensor field K satisfies

$$K(X,JY)+JK(X,Y)=0\tag{2.7}$$

for any $X,Y\in C^\infty(T\tilde{M})$.

Definition 2.8 ([10])**.** A holomorphic statistical manifold $(\tilde{M},J,\tilde{g},\tilde{\nabla})$ is said to be of *constant holomorphic sectional curvature* $\tilde{c}\in\mathbb{R}$ if its statistical curvature tensor field satisfies

$$\begin{aligned}\tilde{S}(X,Y)Z=\frac{\tilde{c}}{4}\big\{&\tilde{g}(Y,Z)X-\tilde{g}(X,Z)Y+\tilde{g}(JY,Z)JX\\&-\tilde{g}(JX,Z)JY-2\tilde{g}(JX,Y)JZ\big\}\end{aligned}\tag{2.8}$$

for any $X,Y,Z\in C^\infty(T\tilde{M})$.

A holomorphic statistical manifold of constant holomorphic sectional curvature can be viewed as a generalization of a complex space form.

Now we review some basics of statistical submanifolds.

Definition 2.9 ([15])**.** Let $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ be a statistical manifold and $f : M \to \tilde{M}$ be an immersion. Denote the tangent mapping and the pullback mapping of f by f_* and f^*, respectively. Define g and ∇ on M by

$$g = f^*\tilde{g}, \quad g\left(\nabla_X Y, Z\right) = \tilde{g}\left(\tilde{\nabla}_X f_* Y, f_* Z\right).$$

Then the pair (∇, g) is a statistical structure on M, which is called the *induced statistical structure* by f from $(\tilde{\nabla}, \tilde{g})$.

Definition 2.10 ([15])**.** Let (M, ∇, g) and $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ be two statistical manifolds. An immersion $f : M \to \tilde{M}$ is called a *statistical immersion* if (∇, g) coincides with the induced statistical structure by f from $(\tilde{\nabla}, \tilde{g})$. Also, (M, ∇, g) is called a *statistical submanifold* of $(\tilde{M}, \tilde{\nabla}, \tilde{g})$.

Similarly to the theory of Riemannian submanifolds, the statistical submanifolds also have the Gauss and Weingarten formulas [20]. Let (M, ∇, g) be a statistical submanifold in $(\tilde{M}, \tilde{\nabla}, \tilde{g})$, denote the set of all smooth tangent vector fields on M by $C^\infty(TM)$, and the set of all smooth normal vector fields on M by $C^\infty(T^\perp M)$; then we have

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \quad \tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y), \\ \tilde{\nabla}_X N &= -A_N X + \nabla^\perp_X N, \quad \tilde{\nabla}^*_X N = -A^*_N X + \nabla^{*\perp}_X N, \end{aligned}$$

where $X, Y \in C^\infty(TM)$, $N \in C^\infty(T^\perp M)$. In the above formulas, h and h^* are the second fundamental forms with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively; A and A^* are the shape operators with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively; $\nabla^\perp$ and $\nabla^{*\perp}$ are the normal connections with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively. Besides, we have the following [21]:

$$h(X, Y) = h(Y, X), \quad h^*(X, Y) = h^*(Y, X), \tag{2.9}$$

$$g(A_N X, Y) = \tilde{g}(h^*(X, Y), N), \quad g(A^*_N X, Y) = \tilde{g}(h(X, Y), N). \tag{2.10}$$

In addition, the statistical submanifolds also have the Gauss equation.

Proposition 2.11 ([10])**.** *Let (M, ∇, g) be a statistical submanifold in $(\tilde{M}, \tilde{\nabla}, \tilde{g})$. For any $X, Y, Z \in C^\infty(TM)$, we have*

$$\begin{aligned} 2\left[\tilde{S}(X, Y)Z\right]^\top = {} & 2S(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ & + A^*_{h^*(X,Z)}Y - A^*_{h^*(Y,Z)}X, \end{aligned} \tag{2.11}$$

where $[\cdot]^\top$ is the tangent component of the vector field "$\cdot$".

Next, we introduce the knowledge of some special statistical submanifolds in holomorphic statistical manifolds.

Definition 2.12 ([10])**.** Let (M, ∇, g) be a statistical submanifold in a holomorphic statistical manifold $(\tilde{M}, J, \tilde{g}, \tilde{\nabla})$. If there exists a differentiable distribution $\mathfrak{D}$ on M satisfying the following two conditions:

(i) $\mathfrak{D}$ is holomorphic, i.e., $J\mathfrak{D}_x = \mathfrak{D}_x$ for each $x \in M$;
(ii) the orthogonal complementary distribution $\mathfrak{D}^\perp$ is totally real, i.e., $J\mathfrak{D}^\perp_x \subset T^\perp_x M$ for each $x \in M$,

then M is called a *CR-submanifold* in $(\tilde{M}, J, \tilde{g}, \tilde{\nabla})$. In particular, if $\mathfrak{D} = TM$, M is called a *holomorphic submanifold*; if $\mathfrak{D}^\perp = TM$, M is called a *totally real submanifold*; and if $\mathfrak{D}^\perp = TM$ and $J\mathfrak{D}^\perp = T^\perp M$, then M is called a *Lagrangian submanifold.*

Obviously, if M is a Lagrangian submanifold in a $2n$-dimensional holomorphic statistical manifold $(\tilde{M}, J, \tilde{g}, \tilde{\nabla})$, then $\dim(M) = n$.

Proposition 2.13 ([10])**.** *Let (M, ∇, g) be a Lagrangian submanifold in a holomorphic statistical manifold $(\tilde{M}, J, \tilde{g}, \tilde{\nabla})$. Then the following formulas hold:*

$$A_{JX}Y = A_{JY}X, \qquad A^*_{JX}Y = A^*_{JY}X \tag{2.12}$$

for any $X, Y \in C^\infty(TM)$.

Let M be a Lagrangian submanifold in a $2n$-dimensional holomorphic statistical manifold $(\tilde{M}, J, \tilde{g}, \tilde{\nabla})$. We may choose a local orthonormal frame $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ on $\tilde{M}$ such that $\{e_1, \ldots, e_n\}$ are tangent to M. Denote the curvature tensor field of ∇^0 by R^0. Then the scalar curvature of M with respect to ∇^0 is given by

$$2\tau^0 = \sum_{1\leqslant i\neq j\leqslant n} g\left(R^0(e_i, e_j)e_j, e_i\right). \tag{2.13}$$

The Ricci curvature with respect to the statistical curvature tensor field S on M is defined as

$$\mathrm{Ric}(Y, Z) = \mathrm{tr}\{X \mapsto S(X, Y)Z\} = \sum_{i=1}^{n} g\left(S(e_i, Y)Z, e_i\right), \tag{2.14}$$

where $X, Y, Z \in C^\infty(TM)$. In addition, the scalar curvature with respect to the statistical curvature tensor field S on M is defined by

$$2\tau = \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i) = \sum_{1\leqslant i\neq j\leqslant n} g\left(S(e_i, e_j)e_j, e_i\right). \tag{2.15}$$

Write $h^k_{ij} = \tilde{g}(h(e_i, e_j), Je_k), h^{*k}_{ij} = \tilde{g}(h^*(e_i, e_j), Je_k)$. By using (2.12), we have

$$h^k_{ij} = h^j_{ik} = h^i_{kj}, \qquad h^{*k}_{ij} = h^{*j}_{ik} = h^{*i}_{kj}. \tag{2.16}$$

In fact, taking $X = e_i, Y = e_k$ in (2.12) and applying (2.10), we obtain

$$h^k_{ij} = \tilde{g}(h(e_i, e_j), Je_k) = \tilde{g}(A^*_{Je_k}e_i, e_j) = \tilde{g}(A^*_{Je_i}e_k, e_j) = \tilde{g}(h(e_k, e_j), Je_i) = h^i_{kj}.$$

In the same way, we can get $h^{*k}_{ij} = h^{*j}_{ik} = h^{*i}_{kj}$.

In addition, the mean curvature vectors of M with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are defined as

$$H = \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{i=1}^{n} h^k_{ii}\right)Je_k, \qquad H^* = \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{i=1}^{n} h^{*k}_{ii}\right)Je_k.$$

The squared norms of the mean curvature vectors H and H^* are respectively given by

$$\|H\|^2 = \frac{1}{n^2}\sum_{k=1}^{n}\left(\sum_{i=1}^{n} h_{ii}^k\right)^2, \qquad \|H^*\|^2 = \frac{1}{n^2}\sum_{k=1}^{n}\left(\sum_{i=1}^{n} h_{ii}^{*k}\right)^2.$$

Finally we recall the optimization technique on Riemannian manifolds as follows. Let M be a submanifold in a Riemannian manifold $(\tilde{M}, \tilde{g})$ and $f : \tilde{M} \to \mathbb{R}$ be a differentiable function. Consider the constrained extremum problem

$$\min_{x\in M} f(x). \tag{2.17}$$

Lemma 2.14 ([18]). *If $x_0 \in M$ is the solution of the problem* (2.17)*, then*

(i) $(\mathrm{grad} f)(x_0) \in T_{x_0}^{\perp} M$;

(ii) *the bilinear form*

$$\alpha : T_{x_0}M \times T_{x_0}M \to \mathbb{R},$$
$$\alpha(X,Y) = \mathrm{Hess}_f(X,Y) + \widetilde{g}\,(h(X,Y), (\mathrm{grad} f)(x_0))$$

is positive semidefinite, where h is the second fundamental form of M and $\mathrm{grad} f$ *is the gradient of f.*

3. Inequalities for the Oprea invariant

Let M be a Lagrangian submanifold in a $2n$-dimensional holomorphic statistical manifold $(\tilde{M}, J, \tilde{g}, \tilde{\nabla})$ of constant holomorphic sectional curvature, and let (∇, g) be the induced statistical structure. Let $x \in M$, L be an l-dimensional ($l \geqslant 2$) subspace of T_xM, and $X \in L$ be a unit tangent vector. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_xM such that $\{e_1, \ldots, e_l\}$ is an orthonormal basis of L with $e_1 = X$. Then the Ricci curvature of L at X with respect to the Levi-Civita connection ∇^0 is defined as

$$\mathrm{Ric}_L^0(X) = \sum_{i=2}^{l} g\left(R^0(X, e_i)e_i, X\right). \tag{3.1}$$

For $l = 2, \ldots, n$, T. Oprea [17] defined an intrinsic invariant δ_l^0 of M with respect to ∇^0 as follows:

$$\delta_l^0(x) = \tau^0 - \frac{1}{l-1}\min_{\substack{X\in L, \|X\|=1\\ L\subset T_xM}} \mathrm{Ric}_L^0(X). \tag{3.2}$$

Nowadays such an invariant is called *Oprea invariant*.

Analogous with (3.1), the Ricci curvature of L at X with respect to ∇ is defined as

$$\mathrm{Ric}_L(X) = \sum_{i=2}^{l} g\left(S(X, e_i)e_i, X\right). \tag{3.3}$$

Further, we can define the Oprea invariant δ_l of M with respect to ∇ as follows:

$$\delta_l(x) = \tau(x) - \frac{1}{l-1}\min_{\substack{X\in L, \|X\|=1\\ L\subset T_xM}} \mathrm{Ric}_L(X). \tag{3.4}$$

Let $\tilde{K}^0$ be the sectional curvature of $\tilde{M}$ with respect to the Levi-Civita connection $\tilde{\nabla}^0$. Write

$$\tilde{a} = \max_{\pi} \tilde{K}^0(\pi), \qquad \tilde{b} = \min_{\pi} \tilde{K}^0(\pi), \tag{3.5}$$

where π represents any 2-dimensional sectional plane in T_xM.

Theorem 3.1. *Let M be a Lagrangian submanifold in a $2n$-dimensional holomorphic statistical manifold $(\tilde{M}, J, \tilde{g}, \tilde{\nabla})$ of constant holomorphic sectional curvature $4\tilde{c}$, and $l \in \{2, \dots, n-1\}$. Then*

$$4\delta_l^0 - 2\delta_l \leqslant 2n(n-1)\tilde{a} - 4\tilde{b} - (n+1)(n-2)\tilde{c} + \alpha(l)\big(\|H\|^2 + \|H^*\|^2\big), \tag{3.6}$$

where

$$\alpha(l) = \frac{n^2\big(9nl - 14l - 10n + 16\big)}{2\big(9nl + 13l - 10n - 14\big)}.$$

Remark 3.1. If we take $\tilde{\nabla} = \tilde{\nabla}^* = \tilde{\nabla}^0$, then according to (2.8), and noting that M is Lagrangian, we get $\tilde{a} = \tilde{b} = \tilde{c}$, $H = H^* = H^0$, and $\delta_l = \delta_l^0$. In this case, for $l = 2$, Theorem 3.1 becomes Theorem 1.1.

Proof. We first review the Gauss equation with respect to $\tilde{\nabla}^0$:

$$\begin{aligned} g(R^0(X,Y)Z,W) &= \tilde{g}(\tilde{R}^0(X,Y)Z,W) + \tilde{g}(h^0(X,W), h^0(Y,Z)) \\ &\quad - \tilde{g}(h^0(X,Z), h^0(Y,W)). \end{aligned} \tag{3.7}$$

From (2.10), (2.11) is equivalent to

$$\begin{aligned} 2g(S(X,Y)Z,W) &= 2\tilde{g}(\tilde{S}(X,Y)Z,W) + \tilde{g}(h(X,W), h^*(Y,Z)) \\ &\quad - \tilde{g}(h^*(X,Z), h(Y,W)) + \tilde{g}(h^*(X,W), h(Y,Z)) \\ &\quad - \tilde{g}(h(X,Z), h^*(Y,W)). \end{aligned} \tag{3.8}$$

Taking $X = W = e_i,\ Y = Z = e_j$ in (3.7) and (3.8), respectively, and summing over $1 \leqslant i \neq j \leqslant n$, we obtain

$$2\tau^0 = \sum_{1\leqslant i\neq j\leqslant n} \tilde{g}\Big(\tilde{R}^0(e_i,e_j)e_j, e_i\Big) + 2\sum_{k=1}^{n}\sum_{1\leqslant i<j\leqslant n}\Big[h_{ii}^{0k}h_{jj}^{0k} - \big(h_{ij}^{0k}\big)^2\Big], \tag{3.9}$$

$$2\tau = n(n-1)\tilde{c} + \sum_{k=1}^{n}\sum_{1\leqslant i<j\leqslant n}\big(h_{ii}^{k}h_{jj}^{*k} + h_{ii}^{*k}h_{jj}^{k} - 2h_{ij}^{k}h_{ij}^{*k}\big). \tag{3.10}$$

Combining (3.9) and (3.10), and using $2h_{ij}^{0k} = h_{ij}^k + h_{ij}^{*k}$, we get

$$\begin{aligned} 4\tau^0 - 2\tau &= 4\sum_{1\leqslant i<j\leqslant n} \tilde{g}\left(\tilde{R}^0(e_i,e_j)e_j, e_i\right) - n(n-1)\tilde{c} \\ &\quad + \sum_{k=1}^{n}\sum_{1\leqslant i<j\leqslant n}\left[h_{ii}^{k}h_{jj}^{k} + h_{ii}^{*k}h_{jj}^{*k} - (h_{ij}^{k})^2 - (h_{ij}^{*k})^2\right]. \end{aligned} \tag{3.11}$$

From (3.1) and (3.3), we get

$$\frac{\mathrm{Ric}_L^0(e_1)}{l-1} = \frac{1}{l-1}\sum_{i=2}^{l}\tilde{g}(\tilde{R}^0(e_1,e_i)e_i,e_1) + \frac{1}{l-1}\sum_{k=1}^{n}\sum_{i=2}^{l}\left[h_{11}^{0k}h_{ii}^{0k} - (h_{1i}^{0k})^2\right], \tag{3.12}$$

$$\frac{\mathrm{Ric}_L(e_1)}{l-1} = \tilde{c} + \frac{1}{2(l-1)} \sum_{k=1}^{n} \sum_{i=2}^{l} \left(h_{11}^k h_{ii}^{*k} + h_{11}^{*k} h_{ii}^k - 2 h_{1i}^k h_{1i}^{*k} \right). \tag{3.13}$$

From (3.12) and (3.13), and using $2h_{ij}^{0k} = h_{ij}^k + h_{ij}^{*k}$,

$$\begin{aligned} &-4\frac{\mathrm{Ric}^0_L(e_1)}{l-1} + 2\frac{\mathrm{Ric}_L(e_1)}{l-1} \\ &\qquad = -\frac{4}{l-1}\sum_{i=2}^{l} \tilde{g}\left(\tilde{R}^0(e_1,e_i)e_i,e_1\right) + 2\tilde{c} \\ &\qquad\quad - \frac{1}{l-1}\sum_{k=1}^{n}\sum_{i=2}^{l}\left[h_{11}^k h_{ii}^k + h_{11}^{*k} h_{ii}^{*k} - \left(h_{1i}^k\right)^2 - \left(h_{1i}^{*k}\right)^2\right]. \end{aligned} \tag{3.14}$$

Combining (3.11) and (3.14), we have

$$\begin{aligned} &4\tau^0 - 2\tau - 4\frac{\mathrm{Ric}^0_L(e_1)}{l-1} + 2\frac{\mathrm{Ric}_L(e_1)}{l-1} \\ &= 4\sum_{1\leqslant i<j\leqslant n} \tilde{g}\left(\tilde{R}^0(e_i,e_j)e_j,e_i\right) - \frac{4}{l-1}\sum_{i=2}^{l}\tilde{g}\left(\tilde{R}^0(e_1,e_i)e_i,e_1\right) - (n^2-n-2)\tilde{c} \\ &\quad + \sum_{k=1}^{n}\left\{\sum_{1\leqslant i<j\leqslant n} h_{ii}^k h_{jj}^k - \frac{1}{l-1}\sum_{i=2}^{l} h_{11}^k h_{ii}^k + \sum_{1\leqslant i<j\leqslant n} h_{ii}^{*k} h_{jj}^{*k} - \frac{1}{l-1}\sum_{i=2}^{l} h_{11}^{*k} h_{ii}^{*k}\right\} \\ &\quad - \sum_{k=1}^{n}\left\{\sum_{1\leqslant i<j\leqslant n}\left[\left(h_{ij}^k\right)^2 + \left(h_{ij}^{*k}\right)^2\right] - \frac{1}{l-1}\sum_{i=2}^{l}\left[\left(h_{1i}^k\right)^2 + \left(h_{1i}^{*k}\right)^2\right]\right\} \\ &\leqslant 4\sum_{1\leqslant i<j\leqslant n} \tilde{g}\left(\tilde{R}^0(e_i,e_j)e_j,e_i\right) - \frac{4}{l-1}\sum_{i=2}^{l}\tilde{g}\left(\tilde{R}^0(e_1,e_i)e_i,e_1\right) - (n^2-n-2)\tilde{c} \\ &\quad + \sum_{k=1}^{n}\left\{\sum_{1\leqslant i<j\leqslant n} h_{ii}^k h_{jj}^k - \frac{1}{l-1}\sum_{i=2}^{l} h_{11}^k h_{ii}^k + \sum_{1\leqslant i<j\leqslant n} h_{ii}^{*k} h_{jj}^{*k} - \frac{1}{l-1}\sum_{i=2}^{l} h_{11}^{*k} h_{ii}^{*k}\right\} \\ &\quad - \sum_{1\leqslant i\neq j\leqslant n}\left[\left(h_{ii}^j\right)^2 + \left(h_{ii}^{*j}\right)^2\right] + \frac{1}{l-1}\sum_{i=2}^{l}\left[\left(h_{11}^i\right)^2 + \left(h_{11}^{*i}\right)^2 + \left(h_{ii}^1\right)^2 + \left(h_{ii}^{*1}\right)^2\right] \\ &= 4\sum_{1\leqslant i<j\leqslant n} \tilde{g}\left(\tilde{R}^0(e_i,e_j)e_j,e_i\right) - \frac{4}{l-1}\sum_{i=2}^{l}\tilde{g}\left(\tilde{R}^0(e_1,e_i)e_i,e_1\right) \\ &\quad - (n^2-n-2)\tilde{c} + P + P^*, \end{aligned} \tag{3.15}$$

where

$$P = \sum_{k=1}^{n} \left\{ \sum_{1 \leqslant i < j \leqslant n} h_{ii}^{k} h_{jj}^{k} - \frac{1}{l-1} \sum_{i=2}^{l} h_{11}^{k} h_{ii}^{k} \right\} - \sum_{1 \leqslant i \neq j \leqslant n} \left(h_{ii}^{j}\right)^{2} + \frac{1}{l-1} \sum_{i=2}^{l} \left[\left(h_{11}^{i}\right)^{2} + \left(h_{ii}^{1}\right)^{2} \right], \tag{3.16}$$

$$P^{*} = \sum_{k=1}^{n} \left\{ \sum_{1 \leqslant i < j \leqslant n} h_{ii}^{*k} h_{jj}^{*k} - \frac{1}{l-1} \sum_{i=2}^{l} h_{11}^{*k} h_{ii}^{*k} \right\} - \sum_{1 \leqslant i \neq j \leqslant n} \left(h_{ii}^{*j}\right)^{2} + \frac{1}{l-1} \sum_{i=2}^{l} \left[\left(h_{11}^{*i}\right)^{2} + \left(h_{ii}^{*1}\right)^{2} \right]. \tag{3.17}$$

For any $r \in \{2, \dots, l\}$, $t \in \{l+1, \dots, n\}$, consider the quadratic forms $f_1, f_r, f_t : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_1(h_{11}^1, \dots, h_{nn}^1) = \sum_{1 \leqslant i < j \leqslant n} h_{ii}^1 h_{jj}^1 - \frac{1}{l-1} \sum_{i=2}^{l} h_{11}^1 h_{ii}^1 - \sum_{i=2}^{n} \left(h_{ii}^1\right)^2 + \frac{1}{l-1} \sum_{i=2}^{l} \left(h_{ii}^1\right)^2,$$

$$f_r(h_{11}^r, \dots, h_{nn}^r) = \sum_{1 \leqslant i < j \leqslant n} h_{ii}^r h_{jj}^r - \frac{1}{l-1} \sum_{i=2}^{l} h_{11}^r h_{ii}^r - \sum_{i \neq r} \left(h_{ii}^r\right)^2 + \frac{1}{l-1} \left(h_{11}^r\right)^2,$$

$$f_t(h_{11}^t, \dots, h_{nn}^t) = \sum_{1 \leqslant i < j \leqslant n} h_{ii}^t h_{jj}^t - \frac{1}{l-1} \sum_{i=2}^{l} h_{11}^t h_{ii}^t - \sum_{i \neq t} \left(h_{ii}^t\right)^2,$$

respectively; then

$$P = f_1 + \sum_{r=2}^{l} f_r + \sum_{t=l+1}^{n} f_t.$$

Firstly, consider the constrained extremum problem

$$\max f_1, \quad \text{subject to } \mathcal{F}_1 : h_{11}^1 + \cdots + h_{nn}^1 = k_1, \tag{3.18}$$

where k_1 is a real constant. The partial derivatives of the function f_1 are

$$\frac{\partial f_1}{\partial h_{11}^1} = \sum_{i \neq 1}^{n} h_{ii}^1 - \frac{1}{l-1} \sum_{i=2}^{l} h_{ii}^1, \tag{3.19}$$

$$\frac{\partial f_1}{\partial h_{22}^1} = \sum_{i \neq 2}^{n} h_{ii}^1 - \frac{1}{l-1} h_{11}^1 - 2h_{22}^1 + \frac{2}{l-1} h_{22}^1, \tag{3.20}$$

$$\cdots$$

$$\frac{\partial f_1}{\partial h_{ll}^1} = \sum_{i \neq l}^{n} h_{ii}^1 - \frac{1}{l-1} h_{11}^1 - 2h_{ll}^1 + \frac{2}{l-1} h_{ll}^1, \tag{3.21}$$

$$\frac{\partial f_1}{\partial h^1_{(l+1)(l+1)}} = \sum_{i\neq l+1}^{n} h^1_{ii} - 2h^1_{(l+1)(l+1)}, \tag{3.22}$$

$$\cdots$$

$$\frac{\partial f_1}{\partial h^1_{nn}} = \sum_{i=1}^{n-1} h^1_{ii} - 2h^1_{nn}. \tag{3.23}$$

For an optimal solution $\left(h^1_{11}, h^1_{22}, \ldots, h^1_{nn}\right)$ of the problem $\max f_1$, by Lemma 2.14, the vector $\mathrm{grad} f_1$ is normal at $\mathcal{F}_1$, i.e., it is collinear with the vector $(1, 1, \ldots, 1)$. From (3.19)–(3.23), it follows that a critical point of the considered problem has the form

$$\left(h^1_{11}, h^1_{22}, \ldots, h^1_{nn}\right) = (2t_1, t_1, \ldots, t_1). \tag{3.24}$$

As $\sum\limits_{i=1}^{n} h^1_{ii} = k_1$, by using (3.24) we have

$$h^1_{11} = 2h^1_{22} = \cdots = 2h^1_{ll} = 2h^1_{(l+1)(l+1)} = \cdots = 2h^1_{nn} = \frac{2}{n+1}k_1. \tag{3.25}$$

We fix an arbitrary point $x \in \mathcal{F}_1$. The bilinear form $\alpha : T_x\mathcal{F}_1 \times T_x\mathcal{F}_1 \to \mathbb{R}$ has the expression

$$\alpha(X, Y) = \mathrm{Hess} f_1(X, Y) + \langle h'(X, Y), (\mathrm{grad} f_1)(x)\rangle,$$

where h' is the second fundamental form of $\mathcal{F}_1$ in $\mathbb{R}^n$ and $\langle\ ,\ \rangle$ is the standard inner product on $\mathbb{R}^n$. In the standard frame of $\mathbb{R}^n$, the Hessian of f_1 has the matrix

$$\begin{pmatrix}
0 & \frac{l-2}{l-1} & \frac{l-2}{l-1} & \ldots & \frac{l-2}{l-1} & 1 & \ldots & 1 \\
\frac{l-2}{l-1} & \frac{4-2l}{l-1} & 1 & \ldots & 1 & 1 & \ldots & 1 \\
\frac{l-2}{l-1} & 1 & \frac{4-2l}{l-1} & \ldots & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{l-2}{l-1} & 1 & 1 & \ldots & \frac{4-2l}{l-1} & 1 & \ldots & 1 \\
1 & 1 & 1 & \ldots & 1 & -2 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \ldots & 1 & 1 & \ldots & -2
\end{pmatrix}.$$

As $\mathcal{F}_1$ is totally geodesic in $\mathbb{R}^n$, considering a vector X tangent to $\mathcal{F}_1$ at an arbitrary point x on $\mathcal{F}_1$, that is, satisfying the relation $\sum\limits_{i=1}^{n} X_i = 0$, we have

$$\begin{aligned}
\alpha(X, X) &= -X_1^2 - \frac{2}{l-1}\left(X_1X_2 + X_1X_3 + \cdots + X_1X_l\right) \\
&\quad + \frac{5-3l}{l-1}\left(X_2^2 + X_3^2 + \cdots + X_l^2\right) - 3\left(X_{l+1}^2 + X_{l+2}^2 + \cdots + X_n^2\right) \\
&= -\frac{1}{l-1}\left[\left(X_1 + X_2\right)^2 + \left(X_1 + X_3\right)^2 + \cdots + \left(X_1 + X_l\right)^2\right] \\
&\quad - \frac{3l-6}{l-1}\left(X_2^2 + X_3^2 + \cdots + X_l^2\right) - 3\left(X_{l+1}^2 + X_{l+2}^2 + \cdots + X_n^2\right) \leqslant 0.
\end{aligned}$$

Consequently, the point $(h^1_{11}, h^1_{22}, \dots, h^1_{nn})$ given by (3.25) is a global maximum point; here we have used Lemma 2.14. Substituting (3.25) into f_1 we have

$$f_1 \leqslant \frac{n-2}{2(n+1)} k_1^2. \tag{3.26}$$

In the same way, for the constants k_r and k_t, consider the constrained extremum problems

$$\max f_r, \quad \text{subject to } \mathcal{F}_r : h^r_{11} + \cdots + h^r_{nn} = k_r,$$
$$\max f_t, \quad \text{subject to } \mathcal{F}_t : h^t_{11} + \cdots + h^t_{nn} = k_t,$$

respectively. Applying Lemma 2.14, we have

$$f_r \leqslant \frac{9nl - 14l - 7n + 6}{2(9nl + 13l - 7n - 15)} k_r^2, \tag{3.27}$$

$$f_t \leqslant \frac{9nl - 14l - 10n + 16}{2(9nl + 13l - 10n - 14)} k_t^2. \tag{3.28}$$

The equality case of (3.27) holds if and only if

$$\begin{aligned} h^r_{11} &= \frac{2}{9} h^r_{22} = \frac{2}{3} h^r_{33} = \cdots = \frac{2}{3} h^r_{ll} \\ &= \frac{6(l-1)}{9l-7} h^r_{(l+1)(l+1)} = \cdots = \frac{6(l-1)}{9l-7} h^r_{nn} = \frac{6(l-1)}{9nl + 13l - 7n - 15} k_r, \end{aligned}$$

and the equality case of (3.28) holds if and only if

$$\begin{aligned} h^t_{11} &= \frac{2(l-1)}{3l-4} h^t_{22} = \cdots = \frac{2(l-1)}{3l-4} h^t_{ll} \\ &= \frac{6(l-1)}{9l-10} h^t_{(l+1)(l+1)} = \cdots = \frac{6(l-1)}{9l-10} h^t_{(n-1)(n-1)} \\ &= \frac{2(l-1)}{9l-10} h^t_{nn} = \frac{6(l-1)}{9nl + 13l - 10n - 14} k_t. \end{aligned}$$

For $l \geqslant 2$,

$$\frac{n-2}{2(n+1)} \leqslant \frac{9nl - 14l - 7n + 6}{2(9nl + 13l - 7n - 15)} \leqslant \frac{9nl - 14l - 10n + 16}{2(9nl + 13l - 10n - 14)},$$

thus we have

$$f_s \leqslant \frac{9nl - 14l - 10n + 16}{2(9nl + 13l - 10n - 14)} k_s^2 \tag{3.29}$$

for any $s \in \{1, \dots, n\}$. Substituting (3.29) into (3.16) yields

$$P \leqslant \frac{n^2(9nl - 14l - 10n + 16)}{2(9nl + 13l - 10n - 14)} \|H\|^2. \tag{3.30}$$

In the same way, we have

$$P^* \leqslant \frac{n^2(9nl - 14l - 10n + 16)}{2(9nl + 13l - 10n - 14)} \|H^*\|^2. \tag{3.31}$$

Substituting (3.30) and (3.31) into (3.15), we get

$$
\begin{aligned}
&4\tau^0 - 2\tau - 4\frac{\mathrm{Ric}^0_L(e_1)}{l-1} + 2\frac{\mathrm{Ric}_L(e_1)}{l-1} \\
&\qquad \leqslant 4\sum_{1\leqslant i<j\leqslant n} \tilde{g}\left(\tilde{R}^0(e_i,e_j)e_j,e_i\right) - \frac{4}{l-1}\sum_{i=2}^{l} \tilde{g}\left(\tilde{R}^0(e_1,e_i)e_i,e_1\right) \\
&\qquad\qquad - (n^2-n-2)\tilde{c} + \frac{n^2(9nl-14l-10n+16)}{2(9nl+13l-10n-14)}\left(\|H\|^2+\|H^*\|^2\right).
\end{aligned}
$$

From (3.5) we have

$$
\begin{aligned}
&4\tau^0 - 2\tau - 4\frac{\mathrm{Ric}^0_L(e_1)}{l-1} + 2\frac{\mathrm{Ric}_L(e_1)}{l-1} \\
&\qquad\qquad \leqslant 2n(n-1)\tilde{a} - 4\tilde{b} - (n^2-n-2)\tilde{c} \\
&\qquad\qquad\qquad + \frac{n^2(9nl-14l-10n+16)}{2(9nl+13l-10n-14)}\left(\|H\|^2+\|H^*\|^2\right).
\end{aligned}
$$

Using (3.2) and (3.4), we derive the desired inequality. □

For the Oprea invariant δ_n, we can still express the corresponding combination of the scalar curvature and the Ricci curvature in terms of the components of the second fundamental form like (3.15) by using the Gauss equation, and then estimate the corresponding quadratic forms. Note that although the quadratic forms are somehow different from the ones in the case $l = 2, \dots, n-1$, we can also estimate them by using Oprea's optimization method. Since the proof is exactly similar to the proof of Theorem 3.1, we omit it here and only state the final result.

Theorem 3.2. *Let M be a Lagrangian submanifold in a $2n$-dimensional holomorphic statistical manifold $(\tilde{M}, J, \tilde{g}, \tilde{\nabla})$ of constant holomorphic sectional curvature $4\tilde{c}$, $n \geqslant 3$. Then*

$$
\begin{aligned}
4\delta^0_n - 2\delta_n &\leqslant 2n(n-1)\tilde{a} - 4\tilde{b} - (n+1)(n-2)\tilde{c} \\
&\quad + \frac{(3n-1)(n-2)n^2}{2(3n+5)(n-1)}\left(\|H\|^2+\|H^*\|^2\right).
\end{aligned}
\tag{3.32}
$$

Remark 3.2. If we take $\tilde{\nabla} = \tilde{\nabla}^* = \tilde{\nabla}^0$, then according to (2.8), and noting that M is Lagrangian, we get $\tilde{a} = \tilde{b} = \tilde{c}$, $H = H^* = H^0$, and $\delta_n = \delta^0_n$. In this case, Theorem 3.2 becomes Theorem 1.2.

4. Inequalities for the Chen invariant

Let M be a Lagrangian submanifold in a $2n$-dimensional holomorphic statistical manifold $(\tilde{M}, J, \tilde{g}, \tilde{\nabla})$ of constant holomorphic sectional curvature $4\tilde{c}$, (∇, g) be the induced statistical structure from $(\tilde{\nabla}, \tilde{g})$, and S be the statistical curvature tensor field of M. Let $x \in M$ and L be an l-dimensional ($l \geqslant 2$) subspace of T_xM. Choose an orthonormal basis $\{e_1, \dots, e_l\}$ of L. The scalar curvature $\tau^0(L)$ of L with respect to ∇^0 is defined as

$$
\tau^0(L) = \sum_{1\leqslant i<j\leqslant l} g\left(R^0(e_i,e_j)e_j,e_i\right). \tag{4.1}
$$

For an integer $k \geqslant 1$, we denote by $\mathcal{S}(n,k)$ the finite set consisting of all k-tuples $(n_1, n_2, \dots, n_k)$ of integers $\geqslant 2$ satisfying

$$n_1 < n, \quad n_1 + n_2 + \cdots + n_k \leqslant n.$$

Let $\mathcal{S}(n) = \bigcup_{k\geqslant 1} \mathcal{S}(n,k)$. For each $(n_1, n_2, \dots, n_k) \in \mathcal{S}(n)$, the Chen invariant $\delta^0(n_1, n_2, \dots, n_k)$ with respect to ∇^0 is defined as (see [7])

$$\delta^0(n_1, n_2, \dots, n_k) = \tau^0 - \inf\left\{\tau^0(L_1) + \tau^0(L_2) + \cdots + \tau^0(L_k)\right\}, \tag{4.2}$$

where $L_1, L_2, \dots, L_k$ run over all k mutually orthogonal subspaces of T_xM such that $\dim L_j = n_j$, $j = 1, \dots, k$.

Analogous with (4.1), the scalar curvature $\tau(L)$ of L with respect to the statistical curvature tensor field S is defined as

$$\tau(L) = \sum_{1\leqslant i<j\leqslant l} g\left(S(e_i, e_j)e_j, e_i\right). \tag{4.3}$$

Further, we can define the Chen invariant $\delta(n_1, n_2, \dots, n_k)$ with respect to ∇ as follows:

$$\delta(n_1, n_2, \dots, n_k) = \tau - \inf\left\{\tau(L_1) + \tau(L_2) + \cdots + \tau(L_k)\right\}. \tag{4.4}$$

For a given $(n_1, \dots, n_k) \in \mathcal{S}(n)$, let $L_1, L_2, \dots, L_k$ be mutually orthogonal subspaces of T_xM with $\dim L_j = n_j$, $j = 1, \dots, k$. We choose an orthonormal basis $\{e_1, \dots, e_n\}$ of T_xM such that

$$L_j = \operatorname{span}\{e_{n_1+\cdots+n_{j-1}+1}, \dots, e_{n_1+\cdots+n_j}\}, \quad j = 1, \dots, k.$$

Put

$$\begin{aligned} \Delta_1 &= \{1, \dots, n_1\}, \\ &\ \ \vdots \\ \Delta_k &= \{n_1 + \cdots + n_{k-1} + 1, \dots, n_1 \cdots + n_k\}, \\ \Delta_{k+1} &= \{n_1 + \cdots + n_k + 1, \dots, n\}. \end{aligned}$$

For simplicity, write

$$N = n_1 + \cdots + n_k.$$

We shall make use of the following convention on the ranges of indices:

$$\begin{aligned} &\alpha_i,\ \beta_i,\ \gamma_i \in \Delta_i; \quad i,\ j \in \{1, \dots, k\}; \\ &r,\ s,\ t \in \Delta_{k+1}; \quad u,\ v \in \{N+2, \dots, n\}; \\ &A,\ B,\ C \in \{1, \dots, n\}. \end{aligned}$$

B. Y. Chen [6] was the first to establish the inequality between the above invariant and the mean curvature for submanifolds in real space forms. After that, B. Y. Chen and F. Dillen [7] also established a similar inequality for Lagrangian submanifolds in complex space forms (see Theorem 1.3). In this section, we generalize Theorem 1.3 to Lagrangian submanifolds in holomorphic statistical manifolds of constant holomorphic sectional curvature.

Theorem 4.1. *Let M be a Lagrangian submanifold in a $2n$-dimensional holomorphic statistical manifold $(\tilde{M}, J, \tilde{g}, \tilde{\nabla})$ of constant holomorphic sectional curvature $4\tilde{c}$. Denote the sectional curvature of $\tilde{M}$ with respect to $\tilde{\nabla}^0$ by $\tilde{K}^0$. Set $a = \max \tilde{K}^0(e_A \wedge Je_B)$ and $b = \min \tilde{K}^0(e_A \wedge Je_B)$. Then*

$$\begin{aligned}
2\delta^0(n_1, \dots, n_k) \leqslant{} & \delta(n_1, \dots, n_k) \\
& + \left[n(n-1)a - \sum_{i=1}^{k} n_i(n_i - 1)b\right] \\
& - \frac{\tilde{c}}{2}\left[n(n-1) - \sum_{i=1}^{k} n_i(n_i-1)\right] \\
& + \frac{n^2\left(n - \sum_{i=1}^{k} n_i + 3k - 1 - 6\sum_{i=1}^{k} \frac{1}{2+n_i}\right)}{4\left(n - \sum_{i=1}^{k} n_i + 3k + 2 - 6\sum_{i=1}^{k} \frac{1}{2+n_i}\right)}\left(\|H\|^2 + \|H^*\|^2\right).
\end{aligned} \tag{4.5}$$

Moreover, the equality holds at a point $x \in M$ if and only if, with respect to a suitable orthonormal basis $\{e_1, \dots, e_n\}$ at x, the second fundamental forms h and h^* take the following forms:

$$\begin{aligned}
&\sum_{\alpha_j} h^{\gamma_i}_{\alpha_j\alpha_j} = 3h^{\gamma_i}_{rr} = 0, \quad \sum_{\alpha_j} h^{*\gamma_i}_{\alpha_j\alpha_j} = 3h^{*\gamma_i}_{rr} = 0, \\
&h^t_{tt} = (2+n_i)h^t_{\alpha_i\alpha_i} = 3h^t_{rr}, \quad h^{*t}_{tt} = (2+n_i)h^{*t}_{\alpha_i\alpha_i} = 3h^{*t}_{rr}, \\
&h^{\alpha_i}_{\alpha_j\alpha_l} = h^{\alpha_j}_{\alpha_i\beta_i} = h^r_{\alpha_i\alpha_j} = h^{\alpha_i}_{st} = h^r_{st} = 0, \\
&h^{*\alpha_i}_{\alpha_j\alpha_l} = h^{*\alpha_j}_{\alpha_i\beta_i} = h^{*r}_{\alpha_i\alpha_j} = h^{*\alpha_i}_{st} = h^{*r}_{st} = 0
\end{aligned}$$

for distinct $i, j, l \in \{1, \dots, k\}$ and distinct $r, s, t \in \Delta_{k+1}$.

Remark 4.1. If we take $\tilde{\nabla} = \tilde{\nabla}^* = \tilde{\nabla}^0$, then according to (2.8), and noting that M is Lagrangian, we get $a = b = \tilde{c}$, $H = H^* = H^0$, and $\delta(n_1, \dots, n_k) = \delta^0(n_1, \dots, n_k)$. In this case, Theorem 4.1 becomes Theorem 1.3.

Proof. By using (2.15), (4.3) and the Gauss equation (2.11), we calculate

$$\begin{aligned}
&2\left[\tau - \sum_{i=1}^{k} \tau(L_i)\right] \\
&= \tilde{c}\left[n(n-1) - \sum_{i=1}^{k} n_i(n_i-1)\right] + \sum_{A}\sum_{r<s}\left[h^A_{rr}h^{*A}_{ss} + h^{*A}_{rr}h^A_{ss} - 2h^A_{rs}h^{*A}_{rs}\right] \\
&\quad + \sum_{A}\sum_{i<j}\sum_{\alpha_i,\alpha_j}\left[h^A_{\alpha_i\alpha_i}h^{*A}_{\alpha_j\alpha_j} + h^{*A}_{\alpha_i\alpha_i}h^A_{\alpha_j\alpha_j} - 2h^A_{\alpha_i\alpha_j}h^{*A}_{\alpha_i\alpha_j}\right] \\
&\quad + \sum_{A}\sum_{i}\sum_{\alpha_i,s}\left[h^A_{\alpha_i\alpha_i}h^{*A}_{ss} + h^{*A}_{\alpha_i\alpha_i}h^A_{ss} - 2h^A_{\alpha_i s}h^{*A}_{\alpha_i s}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{c}\Big[n(n-1) - \sum_{i=1}^{k} n_i(n_i-1)\Big] \\
&\quad + \sum_{A}\sum_{r<s}\Big[h_{rr}^{A}h_{ss}^{*A} + h_{rr}^{*A}h_{ss}^{A}\Big] + \sum_{A}\sum_{i<j}\sum_{\alpha_i,\alpha_j}\Big[h_{\alpha_i\alpha_i}^{A}h_{\alpha_j\alpha_j}^{*A} + h_{\alpha_i\alpha_i}^{*A}h_{\alpha_j\alpha_j}^{A}\Big] \\
&\quad + \sum_{A}\sum_{i}\sum_{\alpha_i,s}\Big[h_{\alpha_i\alpha_i}^{A}h_{ss}^{*A} + h_{\alpha_i\alpha_i}^{*A}h_{ss}^{A}\Big] + \sum_{A}\sum_{r<s}\Big[\big(h_{rs}^{A}\big)^2 + \big(h_{rs}^{*A}\big)^2\Big] \\
&\quad + \sum_{A}\sum_{i<j}\sum_{\alpha_i,\alpha_j}\Big[\big(h_{\alpha_i\alpha_j}^{A}\big)^2 + \big(h_{\alpha_i\alpha_j}^{*A}\big)^2\Big] + \sum_{A}\sum_{i}\sum_{\alpha_i,s}\Big[\big(h_{\alpha_i s}^{A}\big)^2 + \big(h_{\alpha_i s}^{*A}\big)^2\Big] \\
&\quad - 4\sum_{A}\Big[\sum_{r<s}\big(h_{rs}^{0A}\big)^2 + \sum_{i<j}\sum_{\alpha_i,\alpha_j}\big(h_{\alpha_i\alpha_j}^{0A}\big)^2 + \sum_{i}\sum_{\alpha_i,s}\big(h_{\alpha_i s}^{0A}\big)^2\Big] \\
&\geqslant \tilde{c}\Big[n(n-1) - \sum_{i=1}^{k} n_i(n_i-1)\Big] + \sum_{A}\sum_{\alpha_i,s}\Big[h_{\alpha_i\alpha_i}^{A}h_{ss}^{*A} + h_{\alpha_i\alpha_i}^{*A}h_{ss}^{A}\Big] \\
&\quad + \sum_{A}\Big[\sum_{r<s}\big(h_{rr}^{A}h_{ss}^{*A} + h_{rr}^{*A}h_{ss}^{A}\big) + \sum_{i<j}\sum_{\alpha_i,\alpha_j}\big(h_{\alpha_i\alpha_i}^{A}h_{\alpha_j\alpha_j}^{*A} + h_{\alpha_i\alpha_i}^{*A}h_{\alpha_j\alpha_j}^{A}\big)\Big] \\
&\quad + \sum_{B\neq r}\sum_{r}\Big[\big(h_{BB}^{r}\big)^2 + \big(h_{BB}^{*r}\big)^2\Big] + \sum_{i}\sum_{\alpha_i,r}\Big[\big(h_{rr}^{\alpha_i}\big)^2 + \big(h_{rr}^{*\alpha_i}\big)^2\Big] \\
&\quad - 4\sum_{A}\Big[\sum_{r<s}\big(h_{rs}^{0A}\big)^2 + \sum_{i<j}\sum_{\alpha_i,\alpha_j}\big(h_{\alpha_i\alpha_j}^{0A}\big)^2 + \sum_{i}\sum_{\alpha_i,s}\big(h_{\alpha_i s}^{0A}\big)^2\Big].
\end{aligned}
\tag{4.6}
$$

The equality case of inequality (4.6) holds if and only if

$$
\begin{aligned}
h_{\alpha_j\alpha_l}^{\alpha_i} = h_{\alpha_i\beta_i}^{\alpha_j} = h_{\alpha_i\alpha_j}^{r} = h_{st}^{\alpha_i} = h_{st}^{r} = 0, \\
h_{\alpha_j\alpha_l}^{*\alpha_i} = h_{\alpha_i\beta_i}^{*\alpha_j} = h_{\alpha_i\alpha_j}^{*r} = h_{st}^{*\alpha_i} = h_{st}^{*r} = 0
\end{aligned}
$$

for distinct $i, j, l \in \{1, \ldots, k\}$ and distinct $r, s, t \in \Delta_{k+1}$.

For a given $i \in \{1, \ldots, k\}$ and a given $\gamma_i \in \Delta_i$, we have

$$
\begin{aligned}
0 &\leqslant \sum_{j=1}^{k}\sum_{r}\Big(\sum_{\alpha_j} h_{\alpha_j\alpha_j}^{\gamma_i} - 3h_{rr}^{\gamma_i}\Big)^2 + \sum_{j=1}^{k}\sum_{r}\Big(\sum_{\alpha_j} h_{\alpha_j\alpha_j}^{*\gamma_i} - 3h_{rr}^{*\gamma_i}\Big)^2 \\
&\quad + 3\sum_{r<s}\big(h_{rr}^{\gamma_i} - h_{ss}^{\gamma_i}\big)^2 + 3\sum_{r<s}\big(h_{rr}^{*\gamma_i} - h_{ss}^{*\gamma_i}\big)^2 \\
&\quad + 3\sum_{l<j}\Big(\sum_{\alpha_l} h_{\alpha_l\alpha_l}^{\gamma_i} - \sum_{\alpha_j} h_{\alpha_j\alpha_j}^{\gamma_i}\Big)^2 + 3\sum_{l<j}\Big(\sum_{\alpha_l} h_{\alpha_l\alpha_l}^{*\gamma_i} - \sum_{\alpha_j} h_{\alpha_j\alpha_j}^{*\gamma_i}\Big)^2 \\
&= (n - N + 3k - 3)\Big(\sum_{A} h_{AA}^{\gamma_i}\Big)^2 - 2(n - N + 3k)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \left[\sum_{r<s} h_{rr}^{\gamma_i} h_{ss}^{\gamma_i} + \sum_{j} \sum_{\alpha_j, r} h_{\alpha_j\alpha_j}^{\gamma_i} h_{rr}^{\gamma_i} + \sum_{l<j} \sum_{\alpha_l, \alpha_j} h_{\alpha_l\alpha_l}^{\gamma_i} h_{\alpha_j\alpha_j}^{\gamma_i} - \sum_{r} \left(h_{rr}^{\gamma_i}\right)^2 \right] \\
&\quad + (n-N+3k-3) \left(\sum_{A} h_{AA}^{*\gamma_i} \right)^2 - 2(n-N+3k) \\
&\quad \times \left[\sum_{r<s} h_{rr}^{*\gamma_i} h_{ss}^{*\gamma_i} + \sum_{j} \sum_{\alpha_j, r} h_{\alpha_j\alpha_j}^{*\gamma_i} h_{rr}^{*\gamma_i} + \sum_{l<j} \sum_{\alpha_l, \alpha_j} h_{\alpha_l\alpha_l}^{*\gamma_i} h_{\alpha_j\alpha_j}^{*\gamma_i} - \sum_{r} \left(h_{rr}^{*\gamma_i}\right)^2 \right] \\
&= (n-N+3k-3) \left[\left(\sum_{A} h_{AA}^{\gamma_i} \right)^2 + \left(\sum_{A} h_{AA}^{*\gamma_i} \right)^2 \right] \\
&\quad - 8(n-N+3k) \left[\sum_{r<s} h_{rr}^{0\gamma_i} h_{ss}^{0\gamma_i} + \sum_{j} \sum_{\alpha_j, r} h_{\alpha_j\alpha_j}^{0\gamma_i} h_{rr}^{0\gamma_i} + \sum_{l<j} \sum_{\alpha_l, \alpha_j} h_{\alpha_l\alpha_l}^{0\gamma_i} h_{\alpha_j\alpha_j}^{0\gamma_i} \right] \\
&\quad + 2(n-N+3k) \left\{ \sum_{r<s} \left(h_{rr}^{\gamma_i} h_{ss}^{*\gamma_i} + h_{rr}^{*\gamma_i} h_{ss}^{\gamma_i} \right) + \sum_{j} \sum_{\alpha_j, r} \left(h_{\alpha_j\alpha_j}^{\gamma_i} h_{rr}^{*\gamma_i} + h_{\alpha_j\alpha_j}^{*\gamma_i} h_{rr}^{\gamma_i} \right) \right. \\
&\quad \left. + \sum_{l<j} \sum_{\alpha_l, \alpha_j} \left(h_{\alpha_l\alpha_l}^{\gamma_i} h_{\alpha_j\alpha_j}^{*\gamma_i} + h_{\alpha_l\alpha_l}^{*\gamma_i} h_{\alpha_j\alpha_j}^{\gamma_i} \right) + \sum_{r} \left[\left(h_{rr}^{\gamma_i}\right)^2 + \left(h_{rr}^{*\gamma_i}\right)^2 \right] \right\} \\
&= 2(n-N+3k) \left\{ \frac{n-N+3k-3}{2(n-N+3k)} \left[\left(\sum_{A} h_{AA}^{\gamma_i} \right)^2 + \left(\sum_{A} h_{AA}^{*\gamma_i} \right)^2 \right] \right. \\
&\quad \left. - 4 \left[\sum_{r<s} h_{rr}^{0\gamma_i} h_{ss}^{0\gamma_i} + \sum_{j} \sum_{\alpha_j, r} h_{\alpha_j\alpha_j}^{0\gamma_i} h_{rr}^{0\gamma_i} + \sum_{l<j} \sum_{\alpha_l, \alpha_j} h_{\alpha_l\alpha_l}^{0\gamma_i} h_{\alpha_j\alpha_j}^{0\gamma_i} \right] \right\} \\
&\quad + \sum_{r<s} \left(h_{rr}^{\gamma_i} h_{ss}^{*\gamma_i} + h_{rr}^{*\gamma_i} h_{ss}^{\gamma_i} \right) + \sum_{j} \sum_{\alpha_j, r} \left(h_{\alpha_j\alpha_j}^{\gamma_i} h_{rr}^{*\gamma_i} + h_{\alpha_j\alpha_j}^{*\gamma_i} h_{rr}^{\gamma_i} \right) \\
&\quad + \sum_{l<j} \sum_{\alpha_l, \alpha_j} \left(h_{\alpha_l\alpha_l}^{\gamma_i} h_{\alpha_j\alpha_j}^{*\gamma_i} + h_{\alpha_l\alpha_l}^{*\gamma_i} h_{\alpha_j\alpha_j}^{\gamma_i} \right) + \sum_{r} \left[\left(h_{rr}^{\gamma_i}\right)^2 + (h_{rr}^{*\gamma_i})^2 \right]. \qquad (4.7)
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
&\sum_{r<s} \left(h_{rr}^{\gamma_i} h_{ss}^{*\gamma_i} + h_{rr}^{*\gamma_i} h_{ss}^{\gamma_i} \right) + \sum_{j} \sum_{\alpha_j, r} \left(h_{\alpha_j\alpha_j}^{\gamma_i} h_{rr}^{*\gamma_i} + h_{\alpha_j\alpha_j}^{*\gamma_i} h_{rr}^{\gamma_i} \right) \\
&\qquad + \sum_{l<j} \sum_{\alpha_l, \alpha_j} \left(h_{\alpha_l\alpha_l}^{\gamma_i} h_{\alpha_j\alpha_j}^{*\gamma_i} + h_{\alpha_l\alpha_l}^{*\gamma_i} h_{\alpha_j\alpha_j}^{\gamma_i} \right) + \sum_{r} \left[(h_{rr}^{\gamma_i})^2 + \left(h_{rr}^{*\gamma_i}\right)^2 \right] \\
&\qquad \geqslant 4 \left[\sum_{r<s} h_{rr}^{0\gamma_i} h_{ss}^{0\gamma_i} + \sum_{j} \sum_{\alpha_j, r} h_{\alpha_j\alpha_j}^{0\gamma_i} h_{rr}^{0\gamma_i} + \sum_{l<j} \sum_{\alpha_l, \alpha_j} h_{\alpha_l\alpha_l}^{0\gamma_i} h_{\alpha_j\alpha_j}^{0\gamma_i} \right] \\
&\qquad\qquad - \frac{n-N+3k-3}{2(n-N+3k)} \left[\left(\sum_{A} h_{AA}^{\gamma_i} \right)^2 + \left(\sum_{A} h_{AA}^{*\gamma_i} \right)^2 \right], \qquad (4.8)
\end{aligned}
$$

and the equality holds if and only if

$$\sum_{\alpha_j} h^{\gamma_i}_{\alpha_j\alpha_j} = 3h^{\gamma_i}_{rr}, \qquad \sum_{\alpha_j} h^{*\gamma_i}_{\alpha_j\alpha_j} = 3h^{*\gamma_i}_{rr}. \tag{4.9}$$

Since the following inequalities hold (see [7]),

$$\frac{n-N+3k-3}{2(n-N+3k)} < \frac{n-N+k-1}{2(n-N+k+2)} < \frac{n-N+3k-1-6\sum_{i=1}^{k}\frac{1}{2+n_i}}{2\big(n-N+3k+2-6\sum_{i=1}^{k}\frac{1}{2+n_i}\big)},$$

we get from (4.8) that

$$\begin{aligned}
&\sum_{r<s}\left(h^{\gamma_i}_{rr}h^{*\gamma_i}_{ss}+h^{*\gamma_i}_{rr}h^{\gamma_i}_{ss}\right)+\sum_{j}\sum_{\alpha_j,r}\left(h^{\gamma_i}_{\alpha_j\alpha_j}h^{*\gamma_i}_{rr}+h^{*\gamma_i}_{\alpha_j\alpha_j}h^{\gamma_i}_{rr}\right)\\
&\qquad+\sum_{l<j}\sum_{\alpha_l,\alpha_j}\left(h^{\gamma_i}_{\alpha_l\alpha_l}h^{*\gamma_i}_{\alpha_j\alpha_j}+h^{*\gamma_i}_{\alpha_l\alpha_l}h^{\gamma_i}_{\alpha_j\alpha_j}\right)+\sum_{r}\left[\left(h^{\gamma_i}_{rr}\right)^2+(h^{*\gamma_i}_{rr})^2\right]\\
&\qquad\geqslant 4\left[\sum_{r<s}h^{0\gamma_i}_{rr}h^{0\gamma_i}_{ss}+\sum_{j}\sum_{\alpha_j,r}h^{0\gamma_i}_{\alpha_j\alpha_j}h^{0\gamma_i}_{rr}+\sum_{l<j}\sum_{\alpha_l,\alpha_j}h^{0\gamma_i}_{\alpha_l\alpha_l}h^{0\gamma_i}_{\alpha_j\alpha_j}\right]\\
&\quad-\frac{n-N+3k-1-6\sum_{i=1}^{k}\frac{1}{2+n_i}}{2\big(n-N+3k+2-6\sum_{i=1}^{k}\frac{1}{2+n_i}\big)}\left[\left(\sum_{A}h^{\gamma_i}_{AA}\right)^2+\left(\sum_{A}h^{*\gamma_i}_{AA}\right)^2\right],
\end{aligned} \tag{4.10}$$

and the equality holds if and only if, for each $i \in \{1,\dots,k\}$,

$$\sum_{\alpha_j} h^{\gamma_i}_{\alpha_j\alpha_j} = 3h^{\gamma_i}_{rr} = 0, \qquad \sum_{\alpha_j} h^{*\gamma_i}_{\alpha_j\alpha_j} = 3h^{*\gamma_i}_{rr} = 0. \tag{4.11}$$

Let us put

$$\omega = \frac{2}{3}\left(n-N+3k+2-6\sum_{i=1}^{k}\frac{1}{2+n_i}\right). \tag{4.12}$$

Since

$$\sum_{i=1}^{k}\frac{n_i}{2+n_i} = k-\sum_{i=1}^{k}\frac{2}{2+n_i}, \qquad \sum_{j\neq i}\frac{n_j}{2+n_j} = k-\sum_{j=1}^{k}\frac{2}{2+n_j}-\frac{n_i}{2+n_i},$$

for each $t \in \{N+1,\dots,n\}$ we have

$$\begin{aligned}
0 &\leqslant \sum_{i}\sum_{r\neq t}\frac{2+n_i}{3n_i}\left(\sum_{\alpha_i}h^t_{\alpha_i\alpha_i}-\frac{3n_i}{2+n_i}h^t_{rr}\right)^2\\
&\quad+\sum_{i}\sum_{\alpha_i<\beta_i}\frac{\omega}{n_i}\left(h^t_{\alpha_i\alpha_i}-h^t_{\beta_i\beta_i}\right)^2+\sum_{\substack{r<s\\ r,s\neq t}}\left(h^t_{rr}-h^t_{ss}\right)^2\\
&\quad+\sum_{i<j}\left[\frac{\sqrt{(2+n_i)n_j}}{\sqrt{(2+n_j)n_i}}\sum_{\alpha_i}h^t_{\alpha_i\alpha_i}-\frac{\sqrt{(2+n_j)n_i}}{\sqrt{(2+n_i)n_j}}\sum_{\alpha_j}h^t_{\alpha_j\alpha_j}\right]^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{3}\sum_{r\neq t}\left(h^t_{tt}-3h^t_{rr}\right)^2+\sum_i\frac{n_i}{2+n_i}\left(h^t_{tt}-\frac{2+n_i}{n_i}\sum_{\alpha_i}h^t_{\alpha_i\alpha_i}\right)^2\\
&+\sum_i\sum_{r\neq t}\frac{2+n_i}{3n_i}\left(\sum_{\alpha_i}h^{*t}_{\alpha_i\alpha_i}-\frac{3n_i}{2+n_i}h^{*t}_{rr}\right)^2\\
&+\sum_i\sum_{\alpha_i<\beta_i}\frac{\omega}{n_i}\left(h^{*t}_{\alpha_i\alpha_i}-h^{*t}_{\beta_i\beta_i}\right)^2+\sum_{\substack{r<s\\ r,s\neq t}}\left(h^{*t}_{rr}-h^{*t}_{ss}\right)^2\\
&+\sum_{i<j}\left[\frac{\sqrt{(2+n_i)n_j}}{\sqrt{(2+n_j)n_i}}\sum_{\alpha_i}h^{*t}_{\alpha_i\alpha_i}-\frac{\sqrt{(2+n_j)n_i}}{\sqrt{(2+n_i)n_j}}\sum_{\alpha_j}h^{*t}_{\alpha_j\alpha_j}\right]^2\\
&+\frac{1}{3}\sum_{r\neq t}\left(h^{*t}_{tt}-3h^{*t}_{rr}\right)^2+\sum_i\frac{n_i}{2+n_i}\left(h^{*t}_{tt}-\frac{2+n_i}{n_i}\sum_{\alpha_i}h^{*t}_{\alpha_i\alpha_i}\right)^2\\
=&\ \frac{1}{3}\left(n-N+3k-1-\sum_{i=1}^k\frac{6}{2+n_i}\right)\left(\sum_{\alpha_i}h^t_{\alpha_i\alpha_i}\right)^2\\
&+\omega\sum_i\sum_{\alpha_i}\left(h^t_{\alpha_i\alpha_i}\right)^2+\left(n-N+3k+1-\sum_{i=1}^k\frac{6}{2+n_i}\right)\sum_{r\neq t}\left(h^t_{rr}\right)^2\\
&+\frac{1}{3}\left(n-N+3k-1-\sum_{i=1}^k\frac{6}{2+n_i}\right)\left(h^t_{tt}\right)^2-2\sum_i\sum_{\alpha_i,r}h^t_{\alpha_i\alpha_i}h^t_{rr}\\
&-2\sum_{i<j}\sum_{\alpha_i,\alpha_j}h^t_{\alpha_i\alpha_i}h^t_{\alpha_j\alpha_j}-2\sum_{r<s}h^t_{rr}h^t_{ss}\\
&+\frac{1}{3}\left(n-N+3k-1-\sum_{i=1}^k\frac{6}{2+n_i}\right)\left(\sum_{\alpha_i}h^{*t}_{\alpha_i\alpha_i}\right)^2+\omega\sum_i\sum_{\alpha_i}\left(h^{*t}_{\alpha_i\alpha_i}\right)^2\\
&+\left(n-N+3k+1-\sum_{i=1}^k\frac{6}{2+n_i}\right)\sum_{r\neq t}\left(h^{*t}_{rr}\right)^2\\
&+\frac{1}{3}\left(n-N+3k-1-\sum_{i=1}^k\frac{6}{2+n_i}\right)\left(h^{*t}_{tt}\right)^2-2\sum_i\sum_{\alpha_i,r}h^{*t}_{\alpha_i\alpha_i}h^{*t}_{rr}\\
&-2\sum_{i<j}\sum_{\alpha_i,\alpha_j}h^{*t}_{\alpha_i\alpha_i}h^{*t}_{\alpha_j\alpha_j}-2\sum_{r<s}h^{*t}_{rr}h^{*t}_{ss}.
\end{aligned}
$$

From (4.12), this last inequality is equivalent to

$$
\omega\left[\frac{n-N+3k-1-\sum_{i=1}^k\frac{6}{2+n_i}}{2\big(n-N+3k+2-\sum_{i=1}^k\frac{6}{2+n_i}\big)}\left(\sum_{A=1}^n h^t_{AA}\right)^2+\sum_{r\neq t}\left(h^t_{rr}\right)^2\right.
$$

$$+\sum_i\sum_{\alpha_i}\left(h^t_{\alpha_i\alpha_i}\right)^2-\sum_i\sum_{\alpha_i,r}h^t_{\alpha_i\alpha_i}h^t_{rr}-\sum_{i<j}\sum_{\alpha_i,\alpha_j}h^t_{\alpha_i\alpha_i}h^t_{\alpha_j\alpha_j}-\sum_{r<s}h^t_{rr}h^t_{ss}$$
$$+\frac{n-N+3k-1-\sum_{i=1}^k\frac{6}{2+n_i}}{2\big(n-N+3k+2-\sum_{i=1}^k\frac{6}{2+n_i}\big)}\left(\sum_{A=1}^n h^{*t}_{AA}\right)^2+\sum_{r\neq t}\left(h^{*t}_{rr}\right)^2$$
$$+\sum_i\sum_{\alpha_i}\left(h^{*t}_{\alpha_i\alpha_i}\right)^2-\sum_i\sum_{\alpha_i,r}h^{*t}_{\alpha_i\alpha_i}h^{*t}_{rr}$$
$$\left.-\sum_{i<j}\sum_{\alpha_i,\alpha_j}h^{*t}_{\alpha_i\alpha_i}h^{*t}_{\alpha_j\alpha_j}-\sum_{r<s}h^{*t}_{rr}h^{*t}_{ss}\right]\geqslant 0. \tag{4.13}$$

Noting that $\omega>0$, we get from (4.13) that

$$\sum_{r<s}\left(h^t_{rr}h^{*t}_{ss}+h^{*t}_{rr}h^t_{ss}\right)+\sum_i\sum_{\alpha_i,r}\left(h^t_{\alpha_i\alpha_i}h^{*t}_{rr}+h^{*t}_{\alpha_i\alpha_i}h^t_{rr}\right)$$
$$+\sum_{i<j}\sum_{\alpha_i,\alpha_j}\left(h^t_{\alpha_i\alpha_i}h^{*t}_{\alpha_j\alpha_j}+h^{*t}_{\alpha_i\alpha_i}h^t_{\alpha_j\alpha_j}\right)+\sum_{B\neq t}\left[\left(h^t_{BB}\right)^2+(h^{*t}_{BB})^2\right]$$
$$\geqslant 4\left[\sum_{r<s}h^{0t}_{rr}h^{0t}_{ss}+\sum_i\sum_{\alpha_i,r}h^{0t}_{\alpha_i\alpha_i}h^{0t}_{rr}+\sum_{i<j}\sum_{\alpha_i,\alpha_j}h^{0t}_{\alpha_i\alpha_i}h^{0t}_{\alpha_j\alpha_j}\right]$$
$$-\frac{n-N+3k-1-6\sum_{i=1}^k\frac{1}{2+n_i}}{2\left(n-N+3k+2-6\sum_{i=1}^k\frac{1}{2+n_i}\right)}\left[\left(\sum_A h^t_{AA}\right)^2+\left(\sum_A h^{*t}_{AA}\right)^2\right], \tag{4.14}$$

and the equality holds if and only if

$$h^t_{tt}=(2+n_i)h^t_{\alpha_i\alpha_i}=3h^t_{rr},\qquad h^{*t}_{tt}=(2+n_i)h^{*t}_{\alpha_i\alpha_i}=3h^{*t}_{rr}. \tag{4.15}$$

Summing (4.10) and (4.14) yields

$$\sum_A\left[\sum_{r<s}\left(h^A_{rr}h^{*A}_{ss}+h^{*A}_{rr}h^A_{ss}\right)+\sum_i\sum_{\alpha_i,r}\left(h^A_{\alpha_i\alpha_i}h^{*A}_{rr}+h^{*A}_{\alpha_i\alpha_i}h^A_{rr}\right)\right.$$
$$\left.+\sum_{i<j}\sum_{\alpha_i,\alpha_j}\left(h^A_{\alpha_i\alpha_i}h^{*A}_{\alpha_j\alpha_j}+h^{*A}_{\alpha_i\alpha_i}h^A_{\alpha_j\alpha_j}\right)\right]+\sum_t\sum_{B\neq t}\left[\left(h^t_{BB}\right)^2+\left(h^{*t}_{BB}\right)^2\right]$$
$$+\sum_i\sum_{\alpha_i,r}\left[\left(h^{\alpha_i}_{rr}\right)^2+(h^{*\alpha_i}_{rr})^2\right]$$
$$\geqslant 4\sum_A\left[\sum_{r<s}h^{0A}_{rr}h^{0A}_{ss}+\sum_i\sum_{\alpha_i,r}h^{0A}_{\alpha_i\alpha_i}h^{0A}_{rr}+\sum_{i<j}\sum_{\alpha_i,\alpha_j}h^{0A}_{\alpha_i\alpha_i}h^{0A}_{\alpha_j\alpha_j}\right]$$
$$-\frac{n-N+3k-1-6\sum_{i=1}^k\frac{1}{2+n_i}}{2\left(n-N+3k+2-6\sum_{i=1}^k\frac{1}{2+n_i}\right)}\sum_B\left[\left(\sum_A h^B_{AA}\right)^2+\left(\sum_A h^{*B}_{AA}\right)^2\right]$$

$$
\begin{aligned}
&= 4\left[\tau^0 - \sum_{i=1}^{k}\tau^0(L_i) - \sum_{A<B}\tilde{g}(\tilde{R}^0(e_A,e_B)e_B,e_A)\right. \\
&\quad + \sum_i \sum_{\alpha_i<\beta_i}\tilde{g}\left(\tilde{R}^0(e_{\alpha_i},e_{\beta_i})e_{\beta_i},e_{\alpha_i}\right) + \sum_A\sum_{r<s}\left(h_{rs}^{0A}\right)^2 \\
&\quad \left. + \sum_A\sum_{i<j}\sum_{\alpha_i,\alpha_j}\left(h_{\alpha_i\alpha_j}^{0A}\right)^2 + \sum_A\sum_i\sum_{\alpha_i,r}\left(h_{\alpha_i r}^{0A}\right)^2\right] \\
&\quad - \frac{n-N+3k-1-6\sum_{i=1}^{k}\frac{1}{2+n_i}}{2\left(n-N+3k+2-6\sum_{i=1}^{k}\frac{1}{2+n_i}\right)}\sum_B\left[\left(\sum_A h_{AA}^B\right)^2 + \left(\sum_A h_{AA}^{*B}\right)^2\right].
\end{aligned}
\tag{4.16}
$$

Combining (4.6) and (4.16) we have

$$
\begin{aligned}
2\left[\tau - \sum_{i=1}^{k}\tau(L_i)\right] \geqslant{}& \tilde{c}\left[n(n-1) - \sum_{i=1}^{k} n_i(n_i-1)\right] \\
&+ 4\left[\tau^0 - \sum_{i=1}^{k}\tau^0(L_i) - \sum_{A<B}\tilde{g}\left(\tilde{R}^0\left(e_A,e_B\right)e_B,e_A\right)\right. \\
&\left. + \sum_i\sum_{\alpha_i<\beta_i}\tilde{g}\left(\tilde{R}^0\left(e_{\alpha_i},e_{\beta_i}\right)e_{\beta_i},e_{\alpha_i}\right)\right] \\
&- \frac{n^2\left(n-N+3k-1-6\sum_{i=1}^{k}\frac{1}{2+n_i}\right)}{2\left(n-N+3k+2-6\sum_{i=1}^{k}\frac{1}{2+n_i}\right)}\left(\|H\|^2 + \|H^*\|^2\right).
\end{aligned}
$$

According to (4.2) and (4.4), we obtain

$$
\begin{aligned}
2\delta(n_1,\dots,n_k) \geqslant{}& \tilde{c}\left[n(n-1) - \sum_i n_i(n_i-1)\right] \\
&+ 4\left[\delta^0(n_1,\dots,n_k) - \frac{n(n-1)}{2}a + \sum_i\frac{n_i(n_i-1)}{2}b\right] \\
&- \frac{n^2\left(n-N+3k-1-6\sum_{i=1}^{k}\frac{1}{2+n_i}\right)}{2\left(n-N+3k+2-6\sum_{i=1}^{k}\frac{1}{2+n_i}\right)}\left(\|H\|^2 + \|H^*\|^2\right).
\end{aligned}
\tag{4.17}
$$

Simplifying (4.17) yields inequality (4.5).

Equality in (4.5) implies that the inequalities (4.6), (4.10) and (4.14) become equalities. Thus, the second fundamental forms take the desired forms. □

Acknowledgment

We are thankful to the referee for the valuable comments and suggestions which improved the paper.

References

[1] S. Amari, *Differential-Geometrical Methods in Statistics*, Lecture Notes in Statistics 28, Springer-Verlag, New York, 1985. DOI MR Zbl

[2] M. E. Aydin, A. Mihai, and I. Mihai, Some inequalities on submanifolds in statistical manifolds of constant curvature, *Filomat* **29** no. 3 (2015), 465–477. DOI MR Zbl

[3] M. E. Aydin, A. Mihai, and I. Mihai, Generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature, *Bull. Math. Sci.* **7** no. 1 (2017), 155–166. DOI MR Zbl

[4] B.-Y. Chen, A Riemannian invariant and its applications to submanifold theory, *Results Math.* **27** no. 1-2 (1995), 17–26. DOI MR Zbl

[5] B.-Y. Chen, Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions, *Glasg. Math. J.* **41** no. 1 (1999), 33–41. DOI MR Zbl

[6] B.-Y. Chen, Some new obstructions to minimal and Lagrangian isometric immersions, *Japan. J. Math. (N.S.)* **26** no. 1 (2000), 105–127. DOI MR Zbl

[7] B.-Y. Chen and F. Dillen, Optimal general inequalities for Lagrangian submanifolds in complex space forms, *J. Math. Anal. Appl.* **379** no. 1 (2011), 229–239. DOI MR Zbl

[8] S. Decu, S. Haesen, and L. Verstraelen, Optimal inequalities involving Casorati curvatures, *Bull. Transilv. Univ. Braşov Ser. B (N.S.)* **14(49)**, suppl. (2007), 85–93. MR Zbl

[9] H. Furuhata, Hypersurfaces in statistical manifolds, *Differential Geom. Appl.* **27** no. 3 (2009), 420–429. DOI MR Zbl

[10] H. Furuhata and I. Hasegawa, Submanifold theory in holomorphic statistical manifolds, in *Geometry of Cauchy-Riemann Submanifolds*, Springer, Singapore, 2016, pp. 179–215. DOI MR Zbl

[11] H. Furuhata, I. Hasegawa, Y. Okuyama, K. Sato, and M. H. Shahid, Sasakian statistical manifolds, *J. Geom. Phys.* **117** (2017), 179–186. DOI MR Zbl

[12] T. Kurose, Geometry of statistical manifolds, in *Mathematics in the 21st Century*, Nippon Hyoron Sha, 2004, pp. 34–43.

[13] C. W. Lee, D. W. Yoon, and J. W. Lee, A pinching theorem for statistical manifolds with Casorati curvatures, *J. Nonlinear Sci. Appl.* **10** no. 9 (2017), 4908–4914. DOI MR Zbl

[14] H. Matsuzoe, Statistical manifolds and affine differential geometry, in *Probabilistic Approach to Geometry*, Adv. Stud. Pure Math. 57, Math. Soc. Japan, Tokyo, 2010, pp. 303–321. DOI MR Zbl

[15] C. R. Min, S. O. Choe, and Y. H. An, Statistical immersions between statistical manifolds of constant curvature, *Glob. J. Adv. Res. Class. Mod. Geom.* **3** no. 2 (2014), 66–75. MR Available at https://web.archive.org/web/20200716233636/https://geometry-math-journal.ro/pdf/Volume3-Issue2/3.pdf.

[16] T. Oprea, Generalization of Chen's inequality, *Algebras Groups Geom.* **23** no. 1 (2006), 93–103. MR Zbl

[17] T. Oprea, Chen's inequality in the Lagrangian case, *Colloq. Math.* **108** no. 1 (2007), 163–169. DOI MR Zbl

[18] T. Oprea, On a Riemannian invariant of Chen type, *Rocky Mountain J. Math.* **38** no. 2 (2008), 567–581. DOI MR Zbl

[19] H. Shima and K. Yagi, Geometry of Hessian manifolds, *Differential Geom. Appl.* **7** no. 3 (1997), 277–290. DOI MR Zbl

[20] A. N. SIDDIQUI and M. H. SHAHID, On totally real statistical submanifolds, *Filomat* **32** no. 13 (2018), 4473–4483. DOI MR Zbl

[21] P. W. VOS, Fundamental equations for statistical submanifolds with applications to the Bartlett correction, *Ann. Inst. Statist. Math.* **41** no. 3 (1989), 429–450. DOI MR Zbl

[22] K. YANO and M. KON, *CR Submanifolds of Kaehlerian and Sasakian Manifolds*, Progress in Mathematics 30, Birkhäuser, Boston, MA, 1983. MR Zbl

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Received: July 31, 2021
Accepted: May 3, 2022