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NEW CLASSES OF STATISTICAL MANIFOLDS WITH A COMPLEX STRUCTURE

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Abstract. We define new classes of statistical manifolds with a complex structure. Motivation for our work is the classification of almost Hermitian manifolds with respect to the covariant derivative of the almost complex structure, obtained by Gray and Hervella in 1980. Instead of the Levi-Civita connection, we use a statistical one and obtain eight classes of Kähler manifolds with the statistical connection. Besides, we give some properties of tensors constructed from covariant derivative of the almost complex structure with respect to the statistical connection. From the obtained properties, further investigation of statistical manifolds is possible.

1. Introduction

In their 1980 paper [3], Gray and Hervella obtained sixteen classes of almost Hermitian manifolds. Among them, four are basic: they are called nearly Kähler manifolds, almost Kähler manifolds, Hermitian semi-Kähler manifolds and locally conformal Kähler manifolds. The other twelve classes are direct products of the four basic ones and Kähler manifolds. The idea for this type of classification came from a generalization of Kähler manifolds. Since the defining condition for Kähler manifolds is the parallelism of a complex structure J with respect to the Levi-Civita connection ∇^g, a generalizing condition is that J is not parallel. In that sense, the authors in [3] studied the properties of a tensor $g((\nabla^g_X J)Y, Z)$, and obtained the classification accordingly. Here, g denotes the Riemannian metric.

We are interested in statistical manifolds, especially in their complex version called holomorphic statistical manifolds. They were defined by Kurose in 2004, and we will give their precise definition in the next section. Statistical manifolds were defined by Lauritzen in 1987 [5] as Riemannian manifolds (M, g) with a 3-symmetric tensor T on M. Lauritzen's definition came as a generalization of a statistical model with the Fisher metric and the Amari–Chentsov tensor. Since then, statistical manifolds are studied as purely geometrical objects. They are considered as Riemannian manifolds (M, g) equipped with torsion-free affine connections ∇ and ∇^*, which are dual with respect to the metric g.

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Kurose explored the relationship between statistical manifolds and affine geometry, and generalized the famous work of Shima [8] to statistical manifolds [4]. Many results appeared concerning curvature tensors of statistical manifolds. Opozda introduced a new type of sectional curvature which can be observed as an object of differential geometry and of linear algebra at the same time [7]. Furthermore, exploration of statistical submanifolds was given by Furuhata [2], and by the author [6].

For more details on statistical manifolds as objects that arise from information geometry, we refer the reader to [1].

Complex statistical manifolds, i.e., holomorphic statistical manifolds, can be considered as a generalization of Kähler manifolds. On these manifolds the statistical connection ∇ is given as a sum of the Levi-Civita connection ∇^g and a $(1,2)$-tensor K satisfying certain properties. The complex structure J is not parallel with respect to the statistical connection ∇.

The idea of this paper is to find new classes of statistical manifolds with a complex structure J by classifying tensors of the form $g((\nabla_X J)Y, Z)$. Following [3], we obtain four basic classes of statistical manifolds with a complex structure.

Throughout this paper, let M be a C^∞ manifold of dimension $2n \geq 2$, ∇ an affine connection on M, and g a Riemannian metric on M. We denote by $\Gamma(E)$ the set of all the C^∞ sections of a vector bundle $E \to M$, and by ∇^g the Levi-Civita connection on M. In the second section, we review definitions and some properties of statistical manifolds. In Section 3, we observe space of tensors with certain properties in a Euclidean space and give its decomposition. Finally, in Section 4 we define four new classes of statistical manifolds with a complex structure. Our main result is given in Theorem 4.1

2. Statistical manifolds

Statistical manifolds are new objects in differential geometry, arising from information geometry. We give their precise definition here (cf. [2]).

Definition 2.1. (1) A triple (M, ∇, g) is called a *statistical manifold* if

(i) ∇ is a torsion free connection and
(ii) $(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z)$ for $X,Y,Z \in \Gamma(TM)$.

(2) ∇^* is called the *dual connection* of ∇ with respect to g if

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z), \quad X,Y,Z \in \Gamma(TM).$$

From the definition, we can write the difference of the Levi-Civita connection ∇^g and the statistical connection ∇ as

$$K(X,Y) := \nabla - \nabla^g,$$

where the $(1,2)$-tensor K satisfies the conditions

$$K(X,Y) = K(Y,X) \tag{2.1}$$

and

$$g(K(X,Y),Z) = g(Y, K(X,Z)), \quad X,Y,Z \in \Gamma(TM). \tag{2.2}$$

By C we will denote a $(0,3)$-tensor defined by

$$C(X,Y,Z) := g(K(X,Y),Z).$$

For the dual connection ∇^*, we have that

$$\nabla^*_X Y = \nabla^g_X Y - K(X,Y).$$

Furthermore, the following relations hold:

$$\begin{aligned}\nabla_X g(Y,Z) = -2C(X,Y,Z) = -\nabla^*_X g(Y,Z),\\ K(X,Y) = \tfrac{1}{2}(\nabla_X Y - \nabla^*_X Y),\\ \nabla^g_X Y = \tfrac{1}{2}(\nabla_X Y + \nabla^*_X Y),\end{aligned}$$

$X,Y,Z \in \Gamma(TM)$.

Associated to K is the tensor $[K,K] : TM^2 \to L(TM;TM)$ defined by

$$[K,K](X,Y)Z := [K_X, K_Y](Z).$$

We define a $(0,4)$-tensor $[K,K] : TM^4 \to \mathbb{R}_M$ by

$$\begin{aligned}[K,K](X,Y,Z,W) &:= g([K,K](X,Y)W,Z)\\ &= g(K(X,Z),K(Y,W)) - g(K(X,W),K(Y,Z)).\end{aligned}$$

Also, let $K' := JK$ and $[K,K'](X,Y,Z,W) := g([K_X, K'_Y](Z),W)$.

A complex version of statistical manifolds is defined as follows (see [4]).

Definition 2.2. Let (M,J,g) be a Kähler manifold and ∇ an affine connection on M. (M,∇,g,J) is called a *holomorphic statistical manifold* if

(1) (M,∇,g) is a statistical manifold and
(2) $\omega = g(*,J*)$ is a ∇-parallel two-form on M.

On a holomorphic statistical manifold (M,J,g,∇), where $\nabla = \nabla^g + K$, we have a related holomorphic statistical structure,

$$\nabla'_X Y = \nabla^g_X Y + K'.$$

Now let (M,J,g) be a Hermitian manifold, that is, a complex structure J and a Riemannian metric g satisfy $g(JX,JY) = g(X,Y)$. For a statistical connection ∇, we will consider the tensors

$$I(X,Y) := (\nabla_X J)Y, \quad I^* := (\nabla^*_X J)Y$$

and

$$T(X,Y,Z) := g(I(X,Y),Z). \tag{2.3}$$

For the tensors I and I^*, the following relation holds (see [6]):

$$I^*(X,Y) = -I(X,Y). \tag{2.4}$$

By Θ we will denote the corresponding Lee form of T with respect to g; Θ is defined by

$$\Theta(Z) = g^{ij} T(E_i, E_j, Z).$$

The tensors I and T are connected with the tensors K and C by the following relations:

$$I(X,Y) = K(X,JY) - JK(X,Y),$$

$$T(X,Y,Z) = C(X,JY,Z) + C(X,Y,JZ), \tag{2.5}$$
$$C(X,Y,Z) = \tfrac{1}{2}\{T(JY,X,Z) - T(X,Z,JY) - T(Z,X,JY)\}. \tag{2.6}$$

Example 2.3. For a one-form u, we define a totally symmetric tensor $K^u : TM^2 \to TM$ by
$$K^u(X,Y) = u(X)Y + u(Y)X + g(X,Y)U,$$
where U is the dual vector field of u, that is, $g(U,Z) = u(Z)$. Obviously, a connection defined by
$$\nabla^u_X Y := \nabla^g_X Y + K^u(X,Y)$$
is a statistical one. It follows that
$$\begin{aligned} I^u(X,Y) := K^u(X,JY) - JK^u(X,Y) = u(JY)X + g(X,JY)U \\ - u(Y)JX - g(X,Y)JU. \end{aligned}$$
The tensor $T^u(X,Y,Z) := g(I^u(X,Y),Z)$ satisfies the following properties:
$$T^u(X,Y,ZY) = T^u(X,Z,Y) = -T^u(X,JY,JZ).$$

Lemma 2.4. *A Kähler manifold (M,J,g) with an affine connection ∇ is holomorphic statistical if the difference tensor K satisfies the conditions* (2.1)*,* (2.2)*, and*
$$K(X,JY) = -JK(X,Y).$$

The curvature tensor on statistical manifolds is defined in a standard way:
$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$
The corresponding $(0,4)$-tensor is given by
$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

3. Some properties of tensors on statistical manifolds with a complex structure

Lemma 3.1. *Given a Kähler manifold (M,J,g) and a totally symmetric tensor K defining a statistical connection, we have the following properties:*

(1) $T(X,Y,Z) = T(X,Z,Y)$.
(2) $T(X,Y,Z) = T(Y,X,Z)$ *if and only if K defines a holomorphic statistical structure.*
(3) $T(X,JY,JZ) = -T(X,Y,Z)$*, or equivalently* $T(X,JY,Z) = T(X,Y,JZ)$.
(4) $T(X,Y,Z) = 2C(X,Y,JZ)$ *if and only if K defines a holomorphic statistical structure.*
(5) *JK defines a statistical connection if and only if K defines a holomorphic statistical connection.*

Let G denote a $(0,3)$-tensor defined by
$$G(X,Y,Z) := T(Y,X,Z) - T(X,Y,Z).$$

Corollary 3.2. *For the tensor G defined on a Kähler manifold (M,J,g) with a totally symmetric tensor K, the following properties hold:*

(1) $G = 0$ *if and only if M is a holomorphic statistical manifold.*

(2) $G(X, Y, Z) = -G(Y, X, Z)$.

Motivated by the properties of the tensor T^u defined in Section 2, we will observe general tensors having the same properties.

On a Euclidean vector space (V, g) of dimension $2m$ with a complex structure J, we consider the space of tensors

$$\mathcal{V} := \{s \in \otimes^3 V^* : s(X, Y, Z) = s(X, Z, Y) = -s(X, JY, JZ)\}.$$

We define a linear operator $L : \mathcal{V} \to \mathcal{V}$ by

$$Ls(X, Y, Z) = s(Y, Z, X) - s(JX, JY, Z).$$

It is elementary to prove that L takes values on $\mathcal{V}$. Since $LLs = 2Ls$, we conclude that the eigenvalues of L are 0 and 2. That is,

$$s(X, Y, Z) = -s(JX, JY, Z)$$

or

$$s(X, Y, Z) = s(JX, JY, Z).$$

Similarly, we will classify the subspace $\mathcal{V}_2 = \{s \in \mathcal{V} : s(X, Y, Z) = -s(JX, JY, Z)\}$ using a map $L_2 : \mathcal{V}_2 \to \mathcal{V}_2$,

$$L_2 s(X, Y, Z) = s(Y, Z, X) + s(Z, X, Y).$$

From $L_2 L_2 = L_2 s + 2s$, we conclude that the eigenvalues of L_2 are -1 and 2. Therefore, $\mathcal{V}_2$ splits as $\mathcal{V}_2 = \mathcal{W}_1 \oplus \mathcal{W}_2$, where

$$\mathcal{W}_1 = \{s \in \mathcal{V}_2 : s(X, Y, Z) + s(Y, Z, X) + s(Z, X, Y) = 0\}$$

and

$$\mathcal{W}_2 = \{s \in \mathcal{V}_2 : 2s(X, Y, JZ) = s(Y, Z, JX) + s(Z, X, JY)\}.$$

Remark 3.3. $T^u(X, Y, Z) = T^u(JX, JY, Z)$.

Finally, on a Euclidean vector space (V, g) of dimension $2m$ with a complex structure J, we can state the following.

Proposition 3.4. *The space of tensors*

$$\mathcal{V} := \{s \in \otimes^3 V^* : s(X, Y, Z) = s(X, Z, Y) = -s(X, JY, JZ)\}$$

can be decomposed as

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2,$$

where

$$\mathcal{V}_1 = \{s \in \mathcal{V} : s(X, Y, Z) = s(JX, JY, Z)\}$$

and

$$\mathcal{V}_2 = \{s \in \mathcal{V} : s(X, Y, Z) = -s(JX, JY, Z)\}.$$

Proposition 3.5. *The space of tensors*

$$\mathcal{V}_2 = \{s \in \mathcal{V} : s(X, Y, Z) = -s(JX, JY, Z)\}$$

can be decomposed as

$$\mathcal{V}_2 = \mathcal{W}_1 \oplus \mathcal{W}_2,$$

where

$$\mathcal{W}_1 = \{s \in \mathcal{V}_2 : s(X,Y,Z) + s(Y,Z,X) + s(Z,X,Y) = 0\}$$

and

$$\mathcal{W}_2 = \{s \in \mathcal{V}_2 : 2s(X,Y,JZ) = s(Y,Z,JX) + s(Z,X,JY)\}.$$

Theorem 3.6. *The space of tensors*

$$\mathcal{V} := \{s \in \otimes^3 V^* : s(X,Y,Z) = s(X,Z,Y) = -s(X,JY,JZ)\}$$

can be decomposed as

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2.$$

4. Classification of statistical manifolds with a complex structure

At this point we give our main result. Since the tensor T defined on a statistical manifold with a complex structure J belongs to $\mathcal{V}$, we can formulate a classification theorem accordingly.

Theorem 4.1. *Given a Kähler manifold (M,J,g) and a totally symmetric tensor K defining a statistical connection we have the following possible properties of a tensor $T(X,Y,Z)$:*

(T0) $T(X,Y,Z) = 2C(X,Y,JZ)$,
(T1) $T(X,Y,Z) = \frac{1}{2n}\{g(X,Z)\Theta(JY) + g(X,JY)\Theta(Z) - g(JX,Z)\Theta(Y) + g(X,Y)\Theta(JZ)\}$,
(T2) $T(X,Y,Z) - T(JX,JY,Z) = \Theta(Z) = 0$,
(T3) $T(X,Y,Z) + T(Y,Z,X) + T(Z,X,Y) = 0$,
(T4) $2T(X,Y,JZ) = T(Y,Z,JX) + T(Z,X,JY)$.

Moreover, we can decompose the space of tensors T defined by (2.3)*, $\mathcal{T}$, in the following way:*

$$\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3 \oplus \mathcal{T}_4,$$

where $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ and $\mathcal{T}_4$ are subspaces of $\mathcal{T}$ whose elements are tensors satisfying (T1)*,* (T2)*,* (T3) *and* (T4)*, respectively, and $\mathcal{T}_0$ is a class of holomorphic statistical manifolds.*

Naturally, we will say that a Kähler manifold M with a statistical connection belongs to classes $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ and $\mathcal{T}_4$ if the tensor T on M satisfies properties (T1), (T2), (T3) and (T4), respectively. Here we give another characterization of Kähler manifolds in terms of the Kähler form ω. From

$$Zg(X,JY) = Z\omega(X,Y)$$

we get

$$g(X,(\nabla^*_Z J)Y) + g(X,J\nabla^*_Z Y) = \nabla_Z\omega(X,Y) + g(X,J\nabla_Z Y).$$

Now we replace (T1) into the previous equation. Using (2.4), the resulting equation is

$$\begin{aligned} g(X,Z)\Theta(JY) + g(Z,JY)\Theta(X) - g(JZ,X)\Theta(Y) + g(Z,Y)\Theta(JX) \\ = \nabla_Z\omega(X,Y) + 2\omega(X,K(Z,Y)). \quad (4.1) \end{aligned}$$

If we replace X and Y in (4.1), and add the obtained equation with (4.1), we get

$$g(X,Z)\Theta(JY) + g(Z,JX)\Theta(Y) + g(Z,JY)\Theta(X) + g(Z,Y)\Theta(JX) \\ = \omega(X, K(Z,Y)) + \omega(Y, K(Z,X)). \quad (4.2)$$

From (4.1) and (4.2), we conclude the following result.

Theorem 4.2. *On a Kähler manifold (M,J,g) that belongs to a class $\mathcal{T}_1$, the Kähler form ω satisfies*

$$\nabla_Z \omega(X,Y) = G(X,Y,Z), \quad X,Y,Z \in \Gamma(TM).$$

Proposition 4.3. *Let (M,J,g) be a Kähler manifold on which the Kähler form ω satisfies*

$$\nabla_Z \omega(X,Y) = G(X,Y,Z), \quad X,Y,Z \in \Gamma(TM).$$

Then,

$$G(JX,JY,Z) = G(X,Y,Z).$$

Proof. From $Zg(X,JY) = Z\omega(X,Y)$, we get

$$C(Z,Y,JX) = \tfrac{1}{2}\{T(Y,Z,X) + T(Z,Y,X) - T(X,Z,Y)\}.$$

On the other hand, from (2.6),

$$C(Z,Y,JX) = \tfrac{1}{2}\{T(JY,Z,JX) - T(Z,JX,JY) - T(JX,Z,JY)\}.$$

From the last two equations and the properties of tensors G and T, we get the conclusion. □

Lemma 4.4. *Given a Kähler manifold (M,J,g) and a totally symmetric tensor K defining a statistical connection, the following equations are equivalent:*

(1) $T(JX,JY,Z) = -T(X,Y,Z)$,
(2) $T(JX,Y,Z) = T(X,JY,Z)$,
(3) $C(X,JY,JZ) = C(X,Y,Z) + C(JX,JY,Z) + C(JX,Y,JZ)$.

Lemma 4.5. *Given a Kähler manifold (M,J,g) and a totally symmetric tensor K defining a statistical connection, the following equations are equivalent:*

(1) $T(JX,JY,Z) = T(X,Y,Z)$,
(2) $T(JX,Y,Z) = -T(X,JY,Z)$,
(3) $C(X,Y,Z) = C(X,JY,JZ) + C(JX,JY,Z) + C(JX,Y,JZ)$.

Definition 4.6. A statistical connection $\nabla = \nabla^0 + K$ on a Kähler manifold (M,J,g) is called a *statistical almost holomorphic connection* if

$$C(X,Y,Z) = C(X,JY,JZ) + C(JX,JY,Z) + C(JX,Y,JZ)$$

holds for all $X,Y,Z \in \Gamma(TM)$.

Let N denote the Nijenhuis tensor defined using the statistical connection ∇ in the following way:

$$N(X,Y) = (\nabla_X J)JY - (\nabla_Y J)JX + (\nabla_{JX} J)Y - (\nabla_{JY} J)X.$$

Moreover, let

$$N'(X,Y) = (\nabla_X J)JY + (\nabla_Y J)JX + (\nabla_{JX} J)Y + (\nabla_{JY} J)X.$$

We express the tensors N and N' in terms of T in the following way:

$$g(N(X,Y),Z) = T(X,JY,Z) - T(Y,JX,Z) + T(JX,Y,Z) - T(JY,X,Z), \quad (4.3)$$

$$g(N'(X,Y),Z) = T(X,JY,Z) + T(Y,JX,Z) + T(JX,Y,Z) + T(JY,X,Z). \quad (4.4)$$

From (4.3) and (4.4), we obtain

$$T(X,JY,Z) + T(JX,Y,Z) = \tfrac{1}{2}\{g(N(X,Y),Z) + g(N'(X,Y),Z)\}.$$

From (4.3) and (2.5), we obtain

$$g(N(X,Y),Z) = 0.$$

Hence,

$$T(X,JY,Z) + T(JX,Y,Z) = T(Y,JX,Z) + T(JY,X,Z). \quad (4.5)$$

Lemma 4.7. *If the tensor T on a Kähler manifold (M,J,g) satisfies the condition $T(JX,JY,Z) = -T(X,Y,Z)$, then (M,J,∇,g) is a holomorphic statistical manifold.*

Proof. If follows from (4.5), Lemma 4.4, and Lemma 3.1 (2). □

Proposition 4.8. *A Kähler manifold (M,J,g) with a statistical connection $\nabla = \nabla^0 + K$ is an almost holomorphic statistical manifold if and only if $N' \equiv 0$ on M.*

Proof. From (2.5) and (4.4), we conclude the equivalence. □

Finally, we give some characterization of a derivative of tensor T.

Theorem 4.9. *On a Kähler manifold (M,J,g) with a statistical connection ∇, a tensor ∇T satisfies the following identities:*

(1) $(\nabla_X T)(Y,Z,W) - (\nabla_Y T)(X,Z,W) = R(X,Y,JZ,W) - R(X,Y,Z,JW) + [K,I](X,Y,Z,W)$,
(2) $(\nabla_X T)(Y,JZ,W) - (\nabla_X T)(Y,Z,JW) = g((\nabla_Y J)Z,(\nabla_X J)W) - g((\nabla_X J)Z,(\nabla_Y J)W)$,
(3) $(\nabla_X T)(Y,Z,W) = (\nabla_X T)(Y,W,Z)$.

Proof. The assertion (1) follows from the identity for J,

$$(\nabla_X(\nabla_Y J))Z - (\nabla_Y(\nabla_X J))Z - (\nabla_{[X,Y]}J)Z = R(X,Y)JZ - JR(X,Y)Z,$$

the definition of derivative of T, and the properties of a statistical connection ∇. Assertions (2) and (3) follow from the properties of T. □

Proposition 4.10. *Let M be a manifold from the class $\mathcal{T}_3$. The following properties hold:*

(1) $(\nabla_X J)JY + (\nabla_{JY} J)X = (\nabla_Y J)JX + (\nabla_{JX} J)Y$,
(2) $T(JX, Y, Z) + T(JY, Z, X) + T(JZ, X, Y) = 0$,
(3) $(\nabla_W T)(X, Y, Z) + (\nabla_W T)(Y, Z, X) + (\nabla_W T)(Z, X, Y) = 0$.

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