REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 64, No. 1, 2022, Pages 23–48
Published online: March 23, 2022
https://doi.org/10.33044/revuma.2869

THE SPACE OF INFINITE PARTITIONS OF $\mathbb{N}$ AS A TOPOLOGICAL RAMSEY SPACE

JULIÁN C. CANO AND CARLOS A. DI PRISCO

Abstract. The Ramsey theory of the space of equivalence relations with infinite quotients defined on the set $\mathbb{N}$ of natural numbers is an interesting field of research. We view this space as a topological Ramsey space $(\mathcal{E}_\infty, \leq, r)$ and present a game theoretic characterization of the Ramsey property of subsets of $\mathcal{E}_\infty$. We define a notion of coideal and consider the Ramsey property of subsets of $\mathcal{E}_\infty$ localized on a coideal $\mathcal{H} \subseteq \mathcal{E}_\infty$. Conditions a coideal $\mathcal{H}$ should satisfy to make the structure $(\mathcal{E}_\infty, \mathcal{H}, \leq, r)$ a Ramsey space are presented. Forcing notions related to a coideal $\mathcal{H}$ and their main properties are analyzed.

1. Introduction

In this article we consider the Ramsey property of collections of equivalence relations with infinite quotients on the set of natural numbers, continuing the study of the *dual Ramsey theory* initiated by Carlson and Simpson in [1] and continued by Halbeisen, Matet and Todorcevic in [12, 17, 14, 23].

Dual Ramsey theory deals with combinatorial properties of sets of partitions analogous to the Ramsey property of sets of infinite subsets of $\mathbb{N}$ as studied in [9, 6, 20].

Let $\mathcal{E}_\infty$ be the set of all infinite equivalence relations on the set of natural numbers. We identify elements of $\mathcal{E}_\infty$ with partitions of the set $\mathbb{N}$ into infinitely many parts.

For a positive integer k, the collection of all equivalence relations on ω with exactly k equivalence classes is denoted by $\mathcal{E}_k$.

Both $\mathcal{E}_\infty$ and $\mathcal{E}_k$ are subsets of $2^{\omega\times\omega}$, and therefore topological spaces with the inherited topology.

We will view $\mathcal{E}_\infty$ as a topological Ramsey space, and we will use the notation and the following definitions as presented in [23, Chapter 5].

If E is a partition of $\mathbb{N}$ into infinitely many equivalence classes, each class $[x]_E$ has a minimal representative. Let $p(E)$ be the set of minimal representatives and $\{p_n(E) : n \in \mathbb{N}\}$ its increasing enumeration. Note that $p_0(E) = 0$ for every $E \in \mathcal{E}_\infty$. In this fashion, the equivalence classes of E are enumerated by $\{[p_n(E)] : n \in \mathbb{N}\}$.

2020 *Mathematics Subject Classification.* 03E05, 03E02.
Key words and phrases. Ramsey spaces; dual Ramsey property; partitions.

The space $\mathcal{E}_\infty$ is endowed with an order relation $\leq$ defined as follows (see [1, 23]): If $E, F \in \mathcal{E}_\infty$, we write $E \leq F$ if E is coarser than F, that is, if every class of E is a union of classes of F. In this case, $p(E) \subseteq p(F)$. Notice that the converse implication is not true in general.

Given a family $\mathcal{E}$ of partitions of $\mathbb{N}$ and $A \in \mathcal{E}_\infty$, the *restriction* of $\mathcal{E}$ to A is defined by

$$\mathcal{E} \upharpoonright A = \{B \in \mathcal{E} : B \leq A\}.$$

For $E \in \mathcal{E}_\infty$, the nth approximation to E is

$$r_n(E) = E \upharpoonright p_n(E),$$

the restriction of the equivalence relation E to the set

$$p_n(E) = \{0, 1, \ldots, p_n(E) - 1\}.$$

For every $E \in \mathcal{E}_\infty$, $r_0(E) = \emptyset$.

The set of all approximations is denoted by $\mathcal{AE}_\infty$, and consists of all equivalence relations on sets of the form $\{0, 1, \ldots, k\}$ for $k \in \mathbb{N}$.

For every $a \in \mathcal{AE}_\infty$, the length of a, denoted by $|a|$, is the unique $n \in \mathbb{N}$ such that $a = r_n(E)$ for some $E \in \mathcal{E}_\infty$; and the domain of a, denoted by $\mathrm{dom}(a)$, is the integer $p_{|a|}(E) = \{0, 1, \ldots, p_{|a|}(E) - 1\}$, where $E \in \mathcal{E}_\infty$ is such that $a = r_{|a|}(E)$. The function $r \colon \mathbb{N} \times \mathcal{E}_\infty \to \mathcal{AE}_\infty$ is defined by $r(n, E) = r_n(E)$. For $n \in \mathbb{N}$, let $\mathcal{AE}_n = \{a \in \mathcal{AE}_\infty : |a| = n\}$; thus $\mathcal{AE}_\infty = \bigcup_{n \in \mathbb{N}} \mathcal{AE}_n$.

Given an approximation $a \in \mathcal{AE}_\infty$ with $\mathrm{dom}(a) = \{0, 1, \ldots, k\}$ and $|a| = n \leq k$, let $p(a) = \{p_0(a), \ldots, p_{n-1}(a)\}$ be the set of minimal representatives of the equivalence classes of a listed in increasing order. Then, the equivalence classes of a are enumerated by $\{[p_0(a)], \ldots, [p_{n-1}(a)]\}$.

For $a, b \in \mathcal{AE}_\infty$, we write $a \sqsubseteq b$ if there are $m, n \in \mathbb{N}$ with $m \leq n$ such that $a = r_m(E)$ and $b = r_n(E)$ for some $E \in \mathcal{E}_\infty$.

The order relation $\leq$ in $\mathcal{E}_\infty$ allows a finitization $\leq_{\mathrm{fin}}$ on $\mathcal{AE}_\infty$ (see [23]): for $a, b \in \mathcal{AE}_\infty$, $a \leq_{\mathrm{fin}} b$ if $\mathrm{dom}(a) = \mathrm{dom}(b)$ and a is coarser than b.

For $A \in \mathcal{E}_\infty$, let

$$\mathcal{AE}_\infty \upharpoonright A = \{a \in \mathcal{AE}_\infty : \exists n (a \leq_{\mathrm{fin}} r_n(A))\},$$
$$\mathcal{AE}_k \upharpoonright A = \{a \in \mathcal{AE}_k : \exists n (a \leq_{\mathrm{fin}} r_n(A))\}.$$

There is a natural topology on $\mathcal{E}_\infty$, namely, the metrizable topology induced by the product topology of $2^{\omega \times \omega}$ (with the discrete topology on 2).

If $a \in \mathcal{AE}_\infty$, we use $[a]$ to denote the set $\{E \in \mathcal{E}_\infty : r_{|a|}(E) = a\}$. It should be noted that $\{[a] : a \in \mathcal{AE}_\infty\}$ is a basis for the metrizable topology on $\mathcal{E}_\infty$.

We will also consider a finer topology on $\mathcal{E}_\infty$, the Ellentuck topology, namely the topology generated by the basic sets of the form

$$[a, E] = \{X \in \mathcal{E}_\infty : a = r_{|a|}(X) \text{ and } X \leq E\},$$

where $a \in \mathcal{AE}_\infty$ and $E \in \mathcal{E}_\infty$. Thus $[\emptyset, E]$ is just $\mathcal{E}_\infty \upharpoonright E = \{X \in \mathcal{E}_\infty : X \leq E\}$.

Notice that a basic set $[a, E]$ is nonempty if and only if there is some n such that $a \leq_{\mathrm{fin}} r_n(E)$.

We will use the symbol $[n, E]$ to abbreviate $[r_n(E), E]$.

Given a basic set $[a, E]$, let

$$\mathcal{AE}_\infty \restriction [a, E] = \{b \in \mathcal{AE}_\infty : a \sqsubseteq b \text{ and } (\exists n \geq |a|)(\exists A \in [a, E])(b = r_n(A))\}.$$

For $A \in \mathcal{E}_\infty$, a partition $A_* \in \mathcal{E}_\infty$ is called a finite modification of A if A_* is obtained by amalgamating a finite number of classes of A. More formally, given a finite set u of natural numbers, let

$$A_u = \{\cup\{x \in A : x \cap u \neq \emptyset\}\} \cup \{x \in A : x \cap u = \emptyset\}.$$

So, A_* is a finite modification of A if there is a finite collection $\{u_1, \dots, u_k\}$ of pairwise disjoint finite subsets of $\mathbb{N}$, such that $A_* = (\dots((A_{u_1})_{u_2})\dots)_{u_k}$, that is,

$$A_* = \{x \in A : x \cap \bigcup_{i=1}^k u_i = \emptyset\} \cup \bigcup_{i=1}^k \{\cup\{x \in A : x \cap u_i \neq \emptyset\}\}.$$

Matet in [17] defines the quasi-order relation $\leq^*$ on $\mathcal{E}_\infty$ (see also [3, 21]): write $A \leq^* B$, and say that A is *almost below* B (or *almost coarser than* B), if there exists a finite modification A_* of A such that $A_* \leq B$. We notice the following:

Fact 1.1. *Let $A, B \in \mathcal{E}_\infty$ be partitions; then $A \leq^* B$ if and only if there exists $a \in \mathcal{AE}_\infty \restriction A$ such that $[a, A] \subseteq [a, B]$.*

Proof. Suppose $A \leq^* B$ and let $A_* \leq A$ be a finite modification of A such that $A_* \leq B$. If $[p_{n_1}(A)], \dots, [p_{n_k}(A)]$ are classes of the partition A that are amalgamated to obtain A_*, let $a \in \mathcal{AE}_1 \restriction A$ be given by $a = \{0, \dots, p_{n_k+1}(A) - 1\}$; then, $[a, A] = [a, A_*]$ and thus $[a, A] \subseteq [a, B]$.

Conversely, suppose that $[a, A] \subseteq [a, B]$ for some $a \in \mathcal{AE}_\infty \restriction A$, and let $n \in \mathbb{N}$ be such that $a \leq_{\text{fin}} r_n(A)$. Let $A_* \in [a, A]$ be such that A_* is obtained by amalgamating a finite number of classes of A in the same way as a is obtained by amalgamating classes of $r_n(A)$. Then, $A_* \leq A$ is a finite modification of A such that $A_* \in [a, B]$, thus $A_* \leq B$, and therefore $A \leq^* B$. ☐

The collection of all partitions of $\mathbb{N}$ can be viewed as a lattice (see [17, 14]). Call $\mathcal{E}_{\leq\infty}$ the set of all partitions of $\mathbb{N}$; then $\mathcal{E}_{\leq\infty} = \mathcal{E}_\infty \cup \bigcup_{k=1}^\infty \mathcal{E}_k$. The order relation $\leq$ defined on $\mathcal{E}_\infty$ extends naturally to $\mathcal{E}_{\leq\infty}$, and induces the lattice operations $\sqcup$ and $\sqcap$ on $\mathcal{E}_{\leq\infty}$ defined as follows: given E and F in $\mathcal{E}_{\leq\infty}$, $E \sqcup F$ is the coarsest partition that refines both E and F, and $E \sqcap F$ is the finest partition that is below both E and F. The set $\mathcal{E}_{\leq\infty}$ together with the operations $\sqcup$ and $\sqcap$ is a bounded lattice with maximum $\{\{n\} : n \in \mathbb{N}\}$ and minimum $\{\mathbb{N}\}$. This lattice is complete, complemented and non-distributive.

We will study combinatorial properties of subsets of $\mathcal{E}_\infty$, and their relations with topological properties.

One of the first results of the combinatorics of partitions is the dual Ramsey theorem of Carlson and Simpson:

Theorem 1.2 ([1])**.** *For every k, m positive integers, if $\mathcal{E}_k = C_1 \cup \dots \cup C_m$ is a finite partition of $\mathcal{E}_\infty$ where every C_i is Borel, then there exists $X \in \mathcal{E}_\infty$ such that $\mathcal{E}_k \restriction X \subseteq C_i$ for some i.*

A subset $\mathcal{X} \subseteq \mathcal{E}_\infty$ determines a partition of $\mathcal{E}_\infty$ into two parts, so one can ask when there is $X \in \mathcal{E}_\infty$ such that $\mathcal{E}_\infty \restriction X$ is contained in one of these parts. This motivates the definition of the Ramsey property of subsets of $\mathcal{E}_\infty$.

Definition 1.3 (Carlson–Simpson)**.** A set $\mathcal{X} \subseteq \mathcal{E}_\infty$ is *Ramsey* if for every neighborhood $[a, A] \neq \emptyset$ there exists $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$.

A set $\mathcal{X} \subseteq \mathcal{E}_\infty$ is *Ramsey null* if for every neighborhood $[a, A] \neq \emptyset$ there exists $B \in [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.

A set $\mathcal{X} \subseteq \mathcal{E}_\infty$ has the *abstract Baire property* if for every neighborhood $[a, A] \neq \emptyset$ there exists $\emptyset \neq [b, B] \subseteq [a, A]$ such that $[b, B] \subseteq \mathcal{X}$ or $[b, B] \cap \mathcal{X} = \emptyset$.

A set $\mathcal{X} \subseteq \mathcal{E}_\infty$ is *nowhere dense* if for every neighborhood $[a, A] \neq \emptyset$ there exists $\emptyset \neq [b, B] \subseteq [a, A]$ such that $[b, B] \cap \mathcal{X} = \emptyset$.

It follows from the definitions that if a set $\mathcal{X} \subseteq \mathcal{E}_\infty$ is Ramsey then it has the abstract Baire property, and that if it is Ramsey null then it is nowhere dense.

Recall that a subset $\mathcal{X}$ of a topological space has the Baire property if there is an open set $\mathcal{O}$ and a meager set $\mathcal{M}$ such that $\mathcal{X} = \mathcal{O} \bigtriangleup \mathcal{M}$.

It is easy to verify that if a set $\mathcal{X} \subseteq \mathcal{E}_\infty$ has the abstract Baire property then it has the Baire property with respect to the Ellentuck topology. For the space $\mathcal{E}_\infty$ with the Ellentuck topology the converse is also true. This is stated in the next fact.

Fact 1.4. *A set $\mathcal{X} \subseteq \mathcal{E}_\infty$ is meager with respect to the Ellentuck topology if and only if it is nowhere dense with respect to this topology.*

A set $\mathcal{X} \subseteq \mathcal{E}_\infty$ has the abstract Baire property if and only if it has the Baire property with respect to the Ellentuck topology.

Proof. Let $\mathcal{M} \subseteq \mathcal{E}_\infty$ be an Ellentuck-meager set. Then $\mathcal{M} = \bigcup_{i=0}^{\infty} N_i$, where every N_i is nowhere dense. We can assume that for every i, $N_i \subseteq N_{i+1}$. Given a basic set $[a, A] \neq \emptyset$, let $[a_0, A_0] \subseteq [a, A]$ such that $[a_0, A_0] \cap N_0 = \emptyset$. Suppose we have defined $[a, A] \supseteq [a_0, A_0] \supseteq [a_1, A_1] \supseteq \cdots \supseteq [a_n, A_n]$ satisfying that for every $i < n$ and every $b \leq_{\text{fin}} a_i$ with $a = r_{|a|}(b)$, $[b, A_i] \cap N_i = \emptyset$. Now, since N_{n+1} is nowhere dense, we can find $[a_{n+1}, A_{n+1}] \subseteq [a_n, A_n]$ such that for every $b \leq_{\text{fin}} a_n$ with $a = r_{|a|}(b)$, $[b, A_{n+1}] \cap N_{n+1} = \emptyset$. Notice that this can be done since the collection of b satisfying $b \leq_{\text{fin}} a_n$ is finite.

We have thus defined inductively a sequence

$$[a, A] \supseteq [a_0, A_0] \supseteq [a_1, A_1] \supseteq \cdots \supseteq [a_n, A_n] \supseteq \ldots.$$

The limit of this sequence is the unique $A_\infty \in [a, A]$ such that $r_{|a_n|}(A_\infty) = r_{|a_n|}(A_n) = a_n$ for every n.

We claim that $[a, A_\infty] \cap \mathcal{M} = \emptyset$. This follows from the fact that $[a, A_\infty] \subseteq [a_n, A_n]$ for every n. Thus, $\mathcal{M}$ is an Ellentuck-nowhere dense set.

The second part of the statement follows from the first one. $\square$

The following theorem gives a topological characterization of the Ramsey property establishing that a set $\mathcal{X} \subseteq \mathcal{E}_\infty$ is Ramsey if and only if it has the Baire property with respect to the Ellentuck topology, and $\mathcal{X} \subseteq \mathcal{E}_\infty$ is Ramsey null if and only if it is meager with respect to this topology.

Theorem 1.5 ([1], [23]). *The triple $(\mathcal{E}_\infty, \leq, r)$ is a topological Ramsey space, that is, every Baire subset of $\mathcal{E}_\infty$ is Ramsey and every meager subset of $\mathcal{E}_\infty$ is Ramsey null.*

The proof of Theorem 1.5 consists in showing that $(\mathcal{E}_\infty, \leq, r)$ satisfies the axioms A.1 to A.4 of [23, Chapter 5], that we will state below. Then the relation between the Ramsey property and the Baire property follows from the theory of Ramsey spaces developed by Todorcevic in [23]; in particular, Theorem 2.2 stated below.

The following combinatorial principle, corresponding to axiom A.4, is of special interest.

Lemma 1.6 ([23]). *Let $[a, E]$ be a nonempty basic set of $\mathcal{E}_\infty$, let n be the length of a and let $\mathcal{O}$ a family of members of $\mathcal{AE}_\infty$ of length $n+1$. Then there is an $F \in [a, E]$ such that $r_{n+1}[a, F]$ is contained in $\mathcal{O}$ or is disjoint from $\mathcal{O}$.*

The Ramsey property and the abstract Baire property of subsets of $\mathcal{E}_\infty$ can be localized on a collection $\mathcal{H} \subseteq \mathcal{E}_\infty$. We introduce now the corresponding definitions.

Definition 1.7. Given a nonempty family $\mathcal{H} \subseteq \mathcal{E}_\infty$, a set $\mathcal{X} \subseteq \mathcal{E}_\infty$ is said to be *$\mathcal{H}$-Ramsey* if for every nonempty $[a, A]$ with $A \in \mathcal{H}$ there exists $B \in \mathcal{H} \cap [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$.

Likewise, a set $\mathcal{X} \subseteq \mathcal{E}_\infty$ is said to be *$\mathcal{H}$-Ramsey null* if for every nonempty $[a, A]$ with $A \in \mathcal{H}$ there exists $B \in \mathcal{H} \cap [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.

The $\mathcal{H}$-Ramsey property of subsets of $\mathcal{E}_\infty$ is a localized version of the dual completely Ramsey property presented in [1].

The abstract Baire property can be localized on a family $\mathcal{H}$ in a similar way.

Definition 1.8. Given a nonempty family $\mathcal{H} \subseteq \mathcal{E}_\infty$, a set $\mathcal{X} \subseteq \mathcal{E}_\infty$ has the *$\mathcal{H}$-Baire* property if for every basic set $[a, A] \neq \emptyset$ with $A \in \mathcal{H}$ there is $\emptyset \neq [b, B] \subseteq [a, A]$ with $B \in \mathcal{H}$ such that $[b, B] \subseteq \mathcal{X}$ or $[b, B] \cap \mathcal{X} = \emptyset$. If the second possibility always occurs, then $\mathcal{X}$ is said to be *$\mathcal{H}$-nowhere dense.*

Ellentuck shows in [6] that the collection of all subsets of $\mathbb{N}^{[\infty]} = \{A \subseteq \mathbb{N} : A \text{ is infinite}\}$ that are completely Ramsey is exactly the algebra of Baire subsets of $\mathbb{N}^{[\infty]}$ with respect to the Ellentuck topology.

In [20], Mathias introduces the notion of happy family (or selective coideal) of subsets of $\mathbb{N}$ and localizes the notion of completely Ramsey subsets of $\mathbb{N}^{[\infty]}$ to such families. He proves that analytic sets are $\mathcal{U}$-Ramsey when $\mathcal{U}$ is a Ramsey ultrafilter and generalizes this result for arbitrary happy families.

Farah, in [7], gives an answer to the question of Todorcevic about what are the combinatorial properties of a family $\mathcal{H} \subseteq \mathbb{N}^{[\infty]}$ under which Borel subsets of $\mathbb{N}^{[\infty]}$ are $\mathcal{H}$-Ramsey. A condition on $\mathcal{H}$ which is weaker than selectivity — the notion called *semiselectivity* — turned out to be enough. Farah shows that the semiselectivity of $\mathcal{H}$ is enough to make the $\mathcal{H}$-Ramsey property equivalent to the abstract Baire property with respect to $\mathcal{H}$, and also shows that this latter equivalence characterizes semiselectivity on $\mathbb{N}^{[\infty]}$.

In [21], a step toward the understanding of the local Ramsey property within the most general context of topological Ramsey spaces is taken (see also [4]).

In this article, we study the notions of selective and semiselective coideal $\mathcal{H} \subseteq \mathcal{E}_\infty$ as well as conditions for $\mathcal{H}$ that will enable us to make the structure $(\mathcal{E}_\infty, \mathcal{H}, \leq, r)$ a Ramsey space, a notion introduced in [23, Chapter 4]. We study versions of the Ramsey property and the Baire property localized on a coideal of $\mathcal{E}_\infty$, and also study forcing notions related to a coideal $\mathcal{H}$ which will satisfy versions of properties of the corresponding forcing notions in the realm of the space of infinite subsets of $\mathbb{N}$, complementing results from [1, 7, 12, 21, 23].

The article is organized as follows: Section 2 is devoted to stating the properties of the space $\mathcal{E}_\infty$ that make it a topological Ramsey space. In Section 3 we characterize the Ramsey property of subsets of $\mathcal{E}_\infty$ in terms of infinite games. In Sections 4 and 5, we define and prove some facts about selective and semiselective coideals of the topological Ramsey space $\mathcal{E}_\infty$. Section 6 deals with two forcing notions related to coideals on $\mathcal{E}_\infty$.

We thank Stevo Todorcevic for valuable conversations on this topic.

2. $\mathcal{E}_\infty$ as a topological Ramsey space

We have defined the space $\mathcal{E}_\infty$ of infinite partitions of $\mathbb{N}$ and the set of approximations $\mathcal{AE}_\infty$. Each $X \in \mathcal{E}_\infty$ can be identified with the sequence of its approximations $\{r_n(X)\}_{n\in\mathbb{N}}$. Give $\mathcal{AE}_\infty$ the discrete topology, and endow $\mathcal{AE}_\infty^{\mathbb{N}}$ with the product topology; this is the complete metric space of all the sequences of elements of $\mathcal{AE}_\infty$. Then, $\mathcal{E}_\infty$ is a closed subspace of the product space $\mathcal{AE}_\infty^{\mathbb{N}}$.

The set $\mathcal{E}_\infty$ with the order relation $\leq$, the approximation function r, and the relation $\leq_{\text{fin}}$ on $\mathcal{AE}_\infty$ satisfies the following properties (or axioms) presented by Todorcevic in [23] (see also [2]):

(A.1) [**Metrization**]

(A.1.1) For any $A \in \mathcal{E}_\infty$, $r_0(A) = \emptyset$.
(A.1.2) For any $A, B \in \mathcal{E}_\infty$, if $A \neq B$ then $(\exists n)$ $(r_n(A) \neq r_n(B))$.
(A.1.3) If $r_n(A) = r_m(B)$ then $n = m$ and $(\forall i < n)$ $(r_i(A) = r_i(B))$.

(A.2) [**Finitization**]

(A.2.1) $\{b \in \mathcal{AE}_\infty : b \leq_{\text{fin}} a\}$ is finite, for every $a \in \mathcal{AE}_\infty$.
(A.2.2) $A \leq B$ iff $(\forall n)$ $(\exists m)$ $(r_n(A) \leq_{\text{fin}} r_m(B))$.
(A.2.3) If $a \leq_{\text{fin}} b$ and $c \sqsubset a$ then there is $d \sqsubset b$ such that $c \leq_{\text{fin}} d$.

Definition 2.1. Given $A \in \mathcal{E}_\infty$ and $a \in \mathcal{AE}_\infty$, the *depth* of a in A, denoted by $\text{depth}_A(a)$, is the only n such that $a \leq_{\text{fin}} r_n(A)$, if such an n exists; otherwise we say that $\text{depth}_A(a)$ is ∞.

(A.3) [**Amalgamation**] Given $a \in \mathcal{AE}_\infty$ and $A \in \mathcal{E}_\infty$ with $\text{depth}_A(a) = n$, the following holds:

(A.3.1) $(\forall B \in [n, A])$ $([a, B] \neq \emptyset)$.
(A.3.2) $(\forall B \in [a, A])$ $(\exists A' \in [n, A])$ $([a, A'] \subseteq [a, B])$.

The last axiom is the following combinatorial principle stated in Lemma 1.6.

(A.4) [**Pigeonhole Principle for** $\mathcal{E}_\infty$] Given a basic set $[a, E] \neq \emptyset$ of $\mathcal{E}_\infty$, let n be the length of a and $\mathcal{O}$ a family of elements of $\mathcal{AE}_\infty$ of length $n+1$. Then there is an $F \in [a, E]$ such that $r_{n+1}[a, F]$ is contained in $\mathcal{O}$ or is disjoint from $\mathcal{O}$.

These properties make the space $(\mathcal{E}_\infty, \leq, r)$ a topological Ramsey space in the sense of Todorcevic [23, Chapter 5], as stated in Theorem 1.5.

The proof that $\mathcal{E}_\infty$ satisfies A.4 is non-trivial. A proof based on an infinite version of the Hales–Jewett theorem for left variable words is given in [23, Lemma 5.69]. This proof is based on the fact that any end extension $b \in r_{n+1}[a, E]$ of a can be coded with a word in the finite alphabet $L = \{0, \dots, n\}$.

Todorcevic in [23] develops an abstract theory of topological Ramsey spaces, and proves the following.

Theorem 2.2 (Abstract Ellentuck theorem; Todorcevic [23])**.** *Any triple $(\mathcal{R}, \leq, r)$ with $\mathcal{R}$ endowed with the Ellentuck topology and metrically closed in the space of infinite sequences of approximations to elements of $\mathcal{R}$, and satisfying* A.1 *to* A.4*, is a topological Ramsey space, meaning that every Baire set is Ramsey and every meager set is Ramsey null.*

3. A game characterization of the Ramsey property

We describe an infinite game on $\mathcal{E}_\infty$ which will be used to characterize the Ramsey property of subsets of the space of infinite partitions $(\mathcal{E}_\infty, \leq, r)$, following ideas of Kastanas in [16] for the Ellentuck space of infinite subsets of ℕ. Similar ideas for the Ellentuck space can be found in [19] and [5]. Games of this type for the space of partitions have been used in [10] and [14] (see also [13]) for purposes other than characterizing the Ramsey property. In [13, Chapter 12], Halbeisen defines Ramsey partition-families using a similar game.

Given $\mathcal{X} \subseteq \mathcal{E}_\infty$ and $\emptyset \neq [a, A] \subseteq \mathcal{E}_\infty$, we define the game $G_{[a,A]}(\mathcal{X})$ as follows:

Player I plays $A_1 \in [a, A]$; player II answers with $[a_1, B_1]$ such that $a \sqsubset a_1$, $|a_1| = |a| + 1$ and $B_1 \in [a_1, A_1]$; player I plays $A_2 \in [a_1, B_1]$; player II then answers $[a_2, B_2]$ with $a_1 \sqsubset a_2$, $|a_2| = |a_1| + 1$ and $B_2 \in [a_2, A_2]$; player I plays $A_3 \in [a_2, B_2]$, and so on, as shown in the following table:

Player I	A_1		A_2		$\dots$
Player II		$[a_1, B_1]$		$[a_2, B_2]$	$\dots$
Rules	$A_1 \in [a, A]$	$a \sqsubset a_1$, $\lvert a_1\rvert = \lvert a\rvert + 1$, $B_1 \in [a_1, A_1]$	$A_2 \in [a_1, B_1]$	$a_1 \sqsubset a_2$, $\lvert a_2\rvert = \lvert a_1\rvert + 1$, $B_2 \in [a_2, A_2]$	$\dots$

Player I wins the game $G_{[a,A]}(\mathcal{X})$ if the only partition of $\bigcap_{n=1}^{\infty}[a_n]$ belongs to $\mathcal{X}$. Otherwise, player II wins the game.

We say that the game $G_{[a,A]}(\mathcal{X})$ is *determined* if one of the players has a winning strategy. The concept of strategy is well known (see, for example, [15, Chapter 33]) so we do not give a formal definition, but we can say that a strategy is a function that tells the player what to play depending on the previous moves of both players. A strategy is a winning strategy for a player if, using this strategy, the outcome of the game is a win for this player independently of the moves of the other player.

Theorem 3.1. *For all $\mathcal{X} \subseteq \mathcal{E}_\infty$, the following statements are equivalent:*

(a) *$\mathcal{X}$ is Ramsey.*
(b) *For every $[a, A] \neq \emptyset$ the game $G_{[a,A]}(\mathcal{X})$ is determined.*

Proof. This result follows from the propositions below. □

Proposition 3.2. *Let $\mathcal{X} \subseteq \mathcal{E}_\infty$ and $[a, A] \neq \emptyset$ be given. Then, player I has a winning strategy in the game $G_{[a,A]}(\mathcal{X})$ if and only if there is $H \in [a, A]$ such that $[a, H] \subseteq \mathcal{X}$.*

Proof. Suppose there is a partition $H \in [a, A]$ such that $[a, H] \subseteq \mathcal{X}$. Then, it is easy to define a winning strategy for player I: play H in the first move, and then play according to the rules. Indeed, if $\langle H; [a_1, B_1]; A_2; [a_2, B_2]; \ldots \rangle$ is a run of the game, it is clear that player I wins since the only partition of $\bigcap_{n=1}^{\infty} [a_n]$ is in $[a, H]$.

Suppose now that player I has a winning strategy σ in the game $G_{[a,A]}(\mathcal{X})$. Without loss of generality suppose, to simplify the argument, that $a = \emptyset$. We construct recursively a decreasing sequence of partitions $\{A_n\}_{n=1}^{\infty}$ as follows:

Stage 1. Let $A_1 = \sigma\langle\emptyset\rangle$, thus the partition A_1 is the first move of player I following the strategy σ, and $A_1 \leq A$. Let $a_1 = r_1(A_1)$.

Suppose that at stage i for each $1 \leq i \leq n$, we have defined partitions A_i and approximations $a_i = r_i(A_i)$ such that $A_1 \geq \cdots \geq A_n$ and $a_1 \sqsubset \cdots \sqsubset a_n$.

Stage $n + 1$. Consider the finite set $\{d \in \mathcal{AE}_\infty : d \leq_{\mathrm{fin}} a_n\}$. Fix an enumeration of this set as $\{d_{(n,1)}, \ldots, d_{(n,p_n)}\}$ with $d_{(n,p_n)} = r_n(A_n) = a_n$.

We will define partitions $C_{(n,k)} \geq B_{(n,k)} \geq A_{(n,k)}$ for each $k \leq p_n$, such that $C_{(n,k)} \in [a_n]$ and $B_{(n,k)}, A_{(n,k)} \in [d_{(n,k)}]$, putting first $C_{(n,0)} = B_{(n,0)} = A_{(n,0)} = C_{(n,1)} = A_n$.

Inductively, suppose that $C_{(n,k)}$, $B_{(n,k)}$ and $A_{(n,k)}$ have been defined. Let $C_{(n,k+1)} \in [a_n, C_{(n,k)}]$ be such that $[d_{(n,k)}, C_{(n,k+1)}] = [d_{(n,k)}, A_{(n,k)}]$, where $C_{(n,k+1)}$ is obtained by splitting a finite number of classes of $A_{(n,k)}$ in the same way as a_n is obtained from $d_{(n,k)}$ through splits. Let $B_{(n,k+1)} \in [d_{(n,k+1)}, C_{(n,k+1)}]$ such that $B_{(n,k+1)}$ is obtained by amalgamating a finite number of classes of $C_{(n,k+1)}$ in the same way as $d_{(n,k+1)}$ is obtained from a_n through amalgamations.

If $r_m(B_{(n,k+1)}) = d_{(n,k+1)}$ with $m > 0$, then we consider the finite sequence $\{r_j(B_{(n,k+1)})\}_{j<m}$ of proper initial segments of $d_{(n,k+1)}$, and let $\rho[d_{(n,k+1)}]$ be the partial run of the game given by this sequence. Since $\{r_j(B_{(n,k+1)}) : j < m\} \subseteq \bigcup_{i<n} \{d \in \mathcal{AE}_\infty : d \leq_{\mathrm{fin}} a_i\}$, the partial run $\rho[d_{(n,k+1)}]$ was built in previous stages of the game, where the consecutive moves of player II are of the form $[r_j(B_{(n,k+1)}), B_{(u_j,v_j)}]$ for each $j < m$. Therefore, for some $u_m < n$ there is a partition $A_{(u_m,v_m)}$ such that $\sigma\left\langle\rho[d_{(n,k+1)}]\right\rangle = A_{(u_m,v_m)}$, and $B_{(n,k+1)} \leq A_{(u_m,v_m)}$. Put $A_{(n,k+1)} = \sigma\left\langle\rho[d_{(n,k+1)}]; A_{(u_m,v_m)}; [d_{(n,k+1)}, B_{(n,k+1)}]\right\rangle$.

Now, we define the partition $A_{n+1} \leq A_n$ as $A_{n+1} = A_{(n,p_n)}$, and let $a_{n+1} = r_{n+1}(A_{n+1})$; thus $a_n \sqsubset a_{n+1}$, because $d_{(n,p_n)} = a_n$. This way we obtain the desired sequences $A_1 \geq A_2 \geq \cdots$ in $\mathcal{E}_\infty \restriction A$ and $a_1 \sqsubset a_2 \sqsubset \cdots$ in $\mathcal{AE}_\infty$.

Finally, let H be the unique partition of $\bigcap_{n=1}^{\infty}[a_n]$. Then $H \in \mathcal{X}$, since the run of the game $\left\langle A_1; [a_1, B_{(1,p_1)}]; A_2; [a_2, B_{(2,p_2)}]; \ldots\right\rangle$ is obtained by player I using the winning strategy σ.

We now show that $[\emptyset, H] \subseteq \mathcal{X}$. Indeed, if $D \leq H$ then there is a unique increasing sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $r_n(D) \leq_{\text{fin}} r_{t_n}(H)$. This implies that for every $n \in \mathbb{N}$ we have that $r_n(D) = d_{(t_n,k_n)}$ for some $1 \leq k_n \leq p_{t_n}$. Hence, if we consider the run of the game

$$\left\langle\sigma\left\langle\rho[d_{(t_1,k_1)}]\right\rangle; [r_1(D), B_{(t_1,k_1)}]; \sigma\left\langle\rho[d_{(t_2,k_2)}]\right\rangle; [r_2(D), B_{(t_2,k_2)}]; \ldots\right\rangle,$$

then it is clear that D is the unique partition of $\bigcap_{n=1}^{\infty}[r_n(D)]$, and since player I uses the winning strategy σ in this run of the game, we conclude that $D \in \mathcal{X}$. □

The referee has pointed out that Proposition 3.2 follows from the pure decision property of the dual-Mathias forcing (see [13, Theorem 28.2]). The proof we have given does not use forcing.

Lemma 3.3. *Let $[a, B]$ be a nonempty basic neighborhood of $\mathcal{E}_\infty$ with $a \sqsubset B$, and let $f\colon [a, B] \to \mathcal{AE}_{|a|+1}$ and $g\colon [a, B] \to \mathcal{E}_\infty$ be functions such that $a \sqsubset f(A)$ and $g(A) \in [f(A), A]$ for each $A \in [a, B]$. Then, there is some $E_{f,g} \in [a, B]$ with the property that for every $q \in \mathcal{AE}_{|a|+1}$ with $a \sqsubset q$ and $\operatorname{depth}_{E_{f,g}}(q) < \infty$, there is $A \in [a, B]$ such that $f(A) = q$ and $[q, E_{f,g}] \subseteq [q, g(A)]$.*

Proof. Given $\emptyset \neq [a, B] \subseteq \mathcal{E}_\infty$, we construct recursively sequences of partitions $\{A_n\}_{n\in\mathbb{N}}$ and $\{A'_n\}_{n\in\mathbb{N}}$ in $[a, B]$, with $A_0 = A'_0 = B$, such that $A'_{n+1} \leq A_{n+1} \leq A'_n \leq A_n$ and $g(A'_n) = A_{n+1}$ for all n, as follows:

Suppose that we have constructed $A_{k-1} \geq A'_{k-1}$ for each $1 \leq k \leq n$ and put $A_n = g(A'_{n-1})$, thus $A_n \in [f(A'_{n-1}), A'_{n-1}]$. Now, let $A'_n \in [a, A_n]$ such that A'_n is obtained by amalgamating the class $[p_{|a|+1}(A_n)]$ to the class of 0 in A_n. Let $A_{n+1} \in [f(A'_n), A'_n]$ be the partition defined by $A_{n+1} = g(A'_n)$; then $\operatorname{depth}_B(f(A'_{n-1})) < \operatorname{depth}_B(f(A'_n))$.

Let now $E_{f,g} \in [a, B]$ be the partition defined by the following equivalence classes: the first $|a| + 1$ classes of $E_{f,g}$ are the first $|a| + 1$ classes of B, except that the first of these, the class of 0, will be enlarged. For $n > 1$, the class $|a| + n$ of $E_{f,g}$ is the class $|a| + 2$ of A_{n-1}. At the end of the construction, the rest of $\mathbb{N}$ is amalgamated to the class of 0.

Finally, we verify that the partition $E_{f,g}$ has the desired properties. Indeed, if $q \in r_{|a|+1}[a, E_{f,g}]$ then it is clear that $q = f(A'_n)$ for some n. Moreover, since $[f(A'_n), E_{f,g}] \subseteq [f(A'_n), A_{n+1}]$, we conclude that $[q, E_{f,g}] \subseteq [q, g(A'_n)]$. □

Proposition 3.4. *Let $\mathcal{X} \subseteq \mathcal{E}_\infty$ and $[a, M] \neq \emptyset$ be given. Then, player II has a winning strategy in the game $G_{[a,M]}(\mathcal{X})$ if and only if for every $N \in [a, M]$ there is $H \in [a, N]$ such that $[a, H] \cap \mathcal{X} = \emptyset$.*

Proof. Given any $N \in [a, M]$, suppose there is a partition $H \in [a, N]$ such that $[a, H] \cap \mathcal{X} = \emptyset$, and let $a_1 = r_{|a|+1}(H)$. Then, it is easy to define a winning strategy for player II: if N is the first move of player I, play $[a_1, H]$ in the first move, and then play according to the rules. Indeed, if $\langle N; [a_1, H]; A_2; [a_2, B_2]; \ldots\rangle$ is a run of the game, it is clear that player II wins since the only partition of $\bigcap_{n=1}^{\infty}[a_n]$ is in $[a, H]$.

Suppose now that player II has a winning strategy τ in the game $G_{[a,M]}(\mathcal{X})$ and let $N \in [a, M]$ be given. Our goal is to define a winning strategy σ for the player I in the game $G_{[a,N]}(\mathcal{E}_\infty \setminus \mathcal{X})$.

Using Lemma 3.3 iteratively, let us construct recursively a play of the game $G_{[a,N]}(\mathcal{E}_\infty \setminus \mathcal{X})$ with consecutive moves of player II given by a sequence $\{[a_n, B_n]\}_{n=1}^{\infty}$, for which we define sequences of functions $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ as well as sequences of partitions $\{A_n\}_{n=1}^{\infty}$ and $\{E_{f_n,g_n}\}_{n=1}^{\infty}$, such that:

(i) $[f_1(A), g_1(A)] = \tau\langle A\rangle$ for every $A \in [a, N]$.
(ii) $[f_{n+1}(A), g_{n+1}(A)] = \tau\,\langle A_1; [f_1(A_1), g_1(A_1)]; \ldots; A_n; [f_n(A_n), g_n(A_n)]; A\rangle$ for every $A \in [a_n, B_n]$.
(iii) $A_1, E_{f_1,g_1} \in [a, N]$ and $A_{n+1}, E_{f_{n+1},g_{n+1}} \in [a_n, B_n]$.
(iv) $f_n(A_n) = a_n$ and $B_n \in [a_n, E_{f_n,g_n}] \subseteq [a_n, g_n(A_n)]$.

Once we have this, we proceed to define a strategy σ for player I in the game $G_{[a,N]}(\mathcal{E}_\infty \setminus \mathcal{X})$ as follows: play consecutively E_{f_n,g_n} in each of the moves, that is to say, $\sigma\langle\emptyset\rangle = E_{f_1,g_1}$ and $\sigma\,\langle E_{f_1,g_1}; [a_1, B_1]; \ldots; E_{f_n,g_n}; [a_n, B_n]\rangle = E_{f_{n+1},g_{n+1}}$.

Finally, let P be the unique partition of $\bigcap_{n=1}^{\infty}[a_n]$. Then $P \notin \mathcal{X}$, since the run of the game $\langle A_1; [a_1, g_1(A_1)]; A_2; [a_2, g_2(A_2)]; \ldots\rangle$ is obtained by player II using the winning strategy τ in $G_{[a,M]}(\mathcal{X})$. Hence, if we consider the run of the game $\langle E_{f_1,g_1}; [a_1, B_1]; E_{f_2,g_2}; [a_2, B_2]; \ldots\rangle$, which is obtained by player I using the strategy σ in $G_{[a,N]}(\mathcal{E}_\infty \setminus \mathcal{X})$, then we deduce that σ is really a winning strategy for the player I in the game $G_{[a,N]}(\mathcal{E}_\infty \setminus \mathcal{X})$ because $P \in \mathcal{E}_\infty \setminus \mathcal{X}$.

Therefore, by Proposition 3.2, we conclude that there is a partition $H \in [a, N]$ such that $[a, H] \subseteq \mathcal{E}_\infty \setminus \mathcal{X}$ and consequently $[a, H] \cap \mathcal{X} = \emptyset$. ☐

Infinite partitions of $\mathbb{N}$ can be coded by infinite subsets of $\mathbb{N}$ (real numbers), so the games defined in this section can be thought as games where the players play real numbers. By Theorem 3.1, $AD_{\mathbb{R}}$, the axiom of determinacy for real numbers, implies that every $\mathcal{X} \subseteq \mathcal{E}_\infty$ is Ramsey. It is an old open problem if AD, the axiom of determinacy, implies that every subset of $\mathbb{N}^\infty$ is Ramsey.

4. Coideals on $\mathcal{E}_\infty$

A coideal on the set $\mathbb{N}$ is a nonempty collection of infinite subsets of $\mathbb{N}$ whose complement in $\mathcal{P}(\mathbb{N})$ is an ideal of sets. Thus, a coideal on $\mathbb{N}$ is closed under supersets and if $A \cup B$ is in the coideal, then at least one of the sets A and B

belongs to the coideal. We propose now a definition of coideal of the space $\mathcal{E}_\infty$ of infinite partitions of $\mathbb{N}$.

Definition 4.1. A family $\mathcal{H} \subseteq \mathcal{E}_\infty$ is a *coideal* if it satisfies:

(i) $\{\{n\} : n \in \mathbb{N}\} \in \mathcal{H}$.
(ii) Let $A, B \in \mathcal{E}_\infty$. If $A \leq^* B$ and $A \in \mathcal{H}$ then $B \in \mathcal{H}$.
(iii) (A.4 mod $\mathcal{H}$) Let $A \in \mathcal{H}$ and $a \in \mathcal{A}\mathcal{E}_\infty \upharpoonright A$ be given. For all $\mathcal{O} \subseteq \mathcal{A}\mathcal{E}_{|a|+1}$ there exists $B \in [\mathrm{depth}_A(a), A] \cap \mathcal{H}$ such that $r_{|a|+1}[a, B] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, B] \cap \mathcal{O} = \emptyset$.

A family $\mathcal{H} \subseteq \mathcal{E}_\infty$ is said to be *closed under finite changes* if whenever two partitions $X, Y \in \mathcal{E}_\infty$ are such that one of them is obtained from the other by amalgamating a finite number of classes, then $X \in \mathcal{H}$ if and only if $Y \in \mathcal{H}$.

Proposition 4.2. *If $\mathcal{H} \subseteq \mathcal{E}_\infty$ is a coideal, then*

(i) *$\mathcal{H}$ is closed under finite changes.*
(ii) (A.3 mod $\mathcal{H}$) *For all $A \in \mathcal{H}$ and $a \in \mathcal{A}\mathcal{E}_\infty \upharpoonright A$, the following holds:*
  (a) *$[a, B] \cap \mathcal{H} \neq \emptyset$ for all $B \in [\mathrm{depth}_A(a), A] \cap \mathcal{H}$.*
  (b) *If $B \in \mathcal{H} \upharpoonright A$ and $[a, B] \neq \emptyset$ then there exists $A' \in [\mathrm{depth}_A(a), A] \cap \mathcal{H}$ such that $\emptyset \neq [a, A'] \subseteq [a, B]$.*

Proof. Let $\mathcal{H} \subseteq \mathcal{E}_\infty$ be a coideal.

(i) If $A, B \in \mathcal{E}_\infty$ are such that A is a finite modification of B, then $A \leq B$ and $B \leq^* A$. So, $A \in \mathcal{H}$ if and only if $B \in \mathcal{H}$.

(ii) Let $A \in \mathcal{H}$ and let $a \in \mathcal{A}\mathcal{E}_\infty \upharpoonright A$ be such that $\mathrm{depth}_A(a) = n$; so, $a \leq_{\mathrm{fin}} r_n(A)$. Given $B \in [n, A] \cap \mathcal{H}$, let $B_* \in [a, B]$ such that it is obtained by amalgamating a finite number of classes of B the same way as a is obtained from $r_n(A)$ by amalgamating classes. Since $\mathcal{H}$ is closed under finite changes and B_* is a finite modification of B, it follows that $B_* \in \mathcal{H}$, concluding (a).

If $B \in \mathcal{H} \upharpoonright A$ is such that $\mathrm{depth}_B(a) = m$, then $a \leq_{\mathrm{fin}} r_m(B) \leq_{\mathrm{fin}} r_n(A)$. Let $A' \in [n, A]$ be such that $[m, B] = [r_m(B), A']$, where A' is obtained by splitting a finite number of classes of B in the same way as $r_n(A)$ is obtained from $r_m(B)$ by splitting classes. Then $B \leq A'$ and so $A' \in \mathcal{H}$. Since $[r_m(B), A'] = [r_m(B), B]$ and $a \leq_{\mathrm{fin}} r_m(B)$, we have $\emptyset \neq [a, A'] \subseteq [a, B]$, concluding (b). □

Clearly, the space $\mathcal{E}_\infty$ is itself a coideal.

Example 4.3. We give now an example of a coideal $\mathcal{H}$ properly contained in $\mathcal{E}_\infty$. Let $\mathcal{H} \subseteq \mathcal{E}_\infty$ be the collection of all infinite partitions of $\mathbb{N}$ with infinitely many finite classes. To see that $\mathcal{H}$ is a coideal we only indicate how to show that $\mathcal{H}$ satisfies A.4 mod $\mathcal{H}$, since the rest of the clauses of the definition are easily verified.

Let $A \in \mathcal{H}$ and $a = r_n(A)$ be given, and let $\mathcal{O} \subseteq \mathcal{A}\mathcal{E}_{|a|+1}$. Form $B \in [a, A]$ amalgamating to the class of 0 all the infinite classes of A with minimum element greater than $p_n(A)$. Then $B \in [a, A] \cap \mathcal{H}$. As in the proof of the Pigeonhole Principle for $\mathcal{E}_\infty$ ([23, Lemma 5.69]), a partition $F \in [a, B] \cap \mathcal{H}$ can be constructed such that $r_{|a|+1}[a, F]$ is contained in $\mathcal{O}$ or is disjoint from $\mathcal{O}$.

For this coideal, there are partitions $X, Y \notin \mathcal{H}$ such that $X \sqcup Y \in \mathcal{H}$. In fact, let $X \in \mathcal{E}_\infty$ be an infinite partition of $\mathbb{N}$ such that all of its classes are infinite. Let $X = \{x_n : n \in \mathbb{N}\}$ be an enumeration of the classes of X, for example putting x_n equal to the class of $p_n(X)$. For every n let $x_n = \{x_n(i) : i \in \mathbb{N}\}$ be the increasing enumeration of the class x_n. Consider now the partition $Y \in \mathcal{E}_\infty$ defined by $Y = \{y_i : i \in \mathbb{N}\}$ where $y_i = \{x_n(i) : n \in \mathbb{N}\}$. Then, Y is an infinite partition of $\mathbb{N}$ and all of its classes are infinite. Thus, $X, Y \notin \mathcal{H}$, but $X \sqcup Y = \{\{n\} : n \in \mathbb{N}\}$, which is in $\mathcal{H}$ since all of its classes are finite. This explains why we did not define coideals using the operation $\sqcup$ of the lattice of partitions.

On the other hand, observe that if $\mathcal{H} \subseteq \mathcal{E}_\infty$ is a coideal, then the collection

$$\mathcal{G} = \{A \subseteq \mathbb{N} : (\exists X \in \mathcal{H})\,(p(X) \subseteq^* A)\}$$

is a coideal on $\mathbb{N}$. If $A \cup B \in \mathcal{G}$, then there is $X \in \mathcal{H}$ such that $p(X) \subseteq^* A \cup B$. Now, $(p(X) \setminus \{0\}) \subseteq \mathcal{AE}_1$, so by A.4 mod $\mathcal{H}$ there is $Y \in \mathcal{H} \restriction X$ such that $(p(Y) \setminus \{0\}) \subseteq A$ or $(p(Y) \setminus \{0\}) \cap A = \emptyset$. So, $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Halbeisen and Matet have introduced the following notion of filter of partitions (see [17, 18, 12, 14]).

Definition 4.4. A nonempty family of partitions $\mathcal{F} \subseteq \mathcal{E}_\infty$ is a *filter* if it satisfies, for every $X, Y \in \mathcal{E}_\infty$, the following conditions:

(a) If $X \leq Y$ and $X \in \mathcal{F}$, then $Y \in \mathcal{F}$.

(b) $X, Y \in \mathcal{F}$ implies that $X \sqcap Y \in \mathcal{F}$.

The relation between these filters and the coideals defined here is interesting and needs further exploration.

4.1. **Selective coideals.** We start with some considerations about the notation $[a, A]$ used for basic sets. If $A \in \mathcal{E}_\infty$ and $a \in \mathcal{AE}_\infty$, even if $\operatorname{depth}_A(a) = \infty$, we can use the notation $[a, A]$ in case a can be obtained from A by splitting a finite number of classes. So, $[a, A]$ is the collection of all partitions B such that $a = r_n(B)$ for some n and each one of the classes $[p_m(B)]$ of B with $m > n$ is a union of classes of A.

Definition 4.5. Let $[a, A]$ be a nonempty basic set of $\mathcal{E}_\infty$. Let $\mathcal{A} = \{A_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of partitions in $[a, A]$. We say that $B \in [a, A]$ is a *diagonalization* of $\mathcal{A}$, if for every $b \in \mathcal{AE}_\infty \restriction B$ such that $a \sqsubseteq b$, we have $[b, B] \subseteq [b, A_{\operatorname{depth}_A(b)}]$.

Matet in [17] gives a definition of diagonalization of a decreasing sequence of partitions in $\mathcal{E}_\infty$. The two definitions are closely related. If we change slightly our definition using $\operatorname{dom}(b)$ instead of $\operatorname{depth}_A(b)$, the relation becomes clearer: if an element of $\mathcal{E}_\infty$ is a diagonalization of the decreasing sequence $\{A_n\}_{n\in\mathbb{N}}$ in this sense then it is a diagonalization according to the definition given by Matet. Our version is designed to work well in the calculations of the next section.

Definition 4.6. A coideal $\mathcal{H}$ on $\mathcal{E}_\infty$ is *selective* if for every $[a, A] \neq \emptyset$ with $A \in \mathcal{H}$, every decreasing sequence $\mathcal{A} = \{A_n\}_{n\in\mathbb{N}}$ of partitions in $[a, A] \cap \mathcal{H}$ has a diagonalization in $[a, A] \cap \mathcal{H}$.

Proposition 4.7. *The space $\mathcal{E}_\infty$ is a selective coideal.*

Proof. It is enough to prove that $\mathcal{E}_\infty$ admits diagonalizations. Given a nonempty basic set $[a, A]$, let $X_0 \geq X_1 \geq X_2 \geq \dots$ be a decreasing sequence of partitions in $[a, A]$.

We define a diagonalization D in $[a, A]$ for this sequence by describing its classes. The first $|a|+1$ classes of D are the first $|a|+1$ classes of X_0, except that the first class is $[p_0(X_0)] \cup R$ where R is a subset of $\mathbb{N}$ that will be described later. The rest of the classes $[p_i(D)]$ are defined recursively as follows: suppose the class $[p_i(D)]$ has been defined, then $r_i(D)$ is already determined; so, if $n = \operatorname{depth}_A(r_i(D))$, then we take $[p_{i+1}(D)] = [p_{i+1}(X_n)]$. Finally, $R = \mathbb{N} \setminus ([p_0(X_0)] \cup \bigcup_{i=1}^{\infty}[p_i(D)])$ is the set of all integers that do not belong to any of the classes defined. □

The argument of the proof of the previous proposition can be used to prove that the coideal $\mathcal{H}$ in Example 4.3 is in fact a selective coideal on $\mathcal{E}_\infty$.

Indeed, given $[a, A] \neq \emptyset$, if $A_0 \geq A_1 \geq A_2 \geq \cdots$ is any decreasing sequence in $\mathcal{H} \cap [a, A]$, then we consider the sequence $A'_0 \geq A'_1 \geq A'_2 \geq \cdots$ in $\mathcal{H} \cap [a, A]$ defined as follows: for every $n \in \mathbb{N}$, the partition $A'_n \in [a, A_n]$ is obtained from A_n by amalgamating to its first class all the infinite classes of A_n that are above $p_{|a|}(A_n)$, therefore $A'_n \in \mathcal{H}$. Now, we form the partition $D \in \mathcal{H} \cap [a, A]$, requiring that the first $|a|+1$ classes of D, except the first one, are also the first $|a|+1$ classes of A'_0, and the other classes of D are defined as follows: if $[p_i(D)]$ has been built, we take $[p_{i+1}(D)] = [p_{i+1}(A'_m)]$ with $m = \operatorname{depth}_A(r_i(D))$. At last, it should be noted that for every $b \in \mathcal{A}\mathcal{E}_\infty \upharpoonright D$ such that $a \sqsubseteq b$, if $n = \operatorname{depth}_A(b)$ then it is true that $[b, D] \subseteq [b, A'_n] \subseteq [b, A_n]$.

Proposition 4.8. *Let $\mathcal{H}$ be a coideal on $\mathcal{E}_\infty$. Then, $\mathcal{H}$ is selective if and only if it has the following two properties:*

(p) *Given $A \in \mathcal{H}$ and a decreasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of partitions in $\mathcal{H} \upharpoonright A$, there is $B \in \mathcal{H} \upharpoonright A$ such that $B \leq^* A_n$ for each $n \in \mathbb{N}$.*

(q) *Given $X \in \mathcal{H}$ and an infinite partition of $p(X)$ into finite pieces, there is $Y \in \mathcal{H} \upharpoonright X$ such that $p(Y)$ has at most one element in each piece.*

Proof. First of all, by Fact 1.1, it is clear that every selective coideal $\mathcal{H}$ on $\mathcal{E}_\infty$ satisfies (p), so let us check that it also satisfies (q). Given $X \in \mathcal{H}$, let $p(X) = \bigcup_{k\in\mathbb{N}} F_k$ be any infinite partition of $p(X)$ where each F_k is finite. For every $n \in \mathbb{N}$, let I_n be the finite set given by $I_n = \{k \in \mathbb{N} : F_k \cap \{0, \dots, p_n(X)\} \neq \emptyset\}$, and let $X_n \in \mathcal{H} \upharpoonright X$ be the finite modification of X obtained by amalgamating the classes $\{[p_k(X)] : k \in I_n\}$. Therefore, $\{X_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of partitions in $\mathcal{H} \upharpoonright X$, and by the selectivity of $\mathcal{H}$ this sequence admits a diagonalization $Y \in \mathcal{H} \upharpoonright X$ which satisfies $|p(Y) \cap F_k| \leq 1$ for all k.

Conversely, suppose that $\mathcal{H}$ has the properties (p) and (q). Let $[a, A] \neq \emptyset$ be a basic set of $\mathcal{E}_\infty$, and without loss of generality suppose, to simplify the argument, that $a = \emptyset$. If $\mathcal{A} = \{A_n\}_{n\in\mathbb{N}}$ is any decreasing sequence of partitions in $\mathcal{H} \upharpoonright A$, then by the property (p), there is a partition $B \in \mathcal{H} \upharpoonright A$ such that $B \leq^* A_n$ for each n.

Define a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ by $n_0 = 0$ and $n_{k+1} = \min\{i \in \mathbb{N} : B/i \leq A_{n_k}\} + (k+1)$, where $B/i \leq A_{n_k}$ means that every class $[p_m(B)]$ of B with $m \geq i$ is a union of classes of A_{n_k}. For every $k \in \mathbb{N}$, let F_k be the finite set given by $F_k = \{p_i(B) : i \in [n_k, n_{k+1})\}$, and consider now the infinite partition $p(B) = \bigcup_{k\in\mathbb{N}} F_k$. Then by the property (q), there is a partition $C \in \mathcal{H} \restriction B$ such that $|p(C) \cap F_k| \leq 1$ for all k.

Finally, let $\mathcal{O} \subseteq \mathcal{AE}_1 \restriction C$ be the collection of 1-approximations of C with domain in some set of the form F_{2k}, so that

$$\mathcal{O} = \{p_n(C) : n > 0 \text{ and } (\exists k)\, p_n(C) \in \{p_i(B) : i \in [n_{2k}, n_{2k+1})\}\}.$$

By A.4 mod $\mathcal{H}$, there is a partition $D \in \mathcal{H} \restriction C$ such that either $(p(D) \setminus \{0\}) \subseteq \mathcal{O}$ or $(p(D) \setminus \{0\}) \cap \mathcal{O} = \emptyset$. In any case, for every $b \in \mathcal{AE}_\infty \restriction D$ with $\operatorname{depth}_A(b) = n$ it is true that $[b, D] \subseteq [b, A_n]$. Thus, $\mathcal{H}$ is a selective coideal on $\mathcal{E}_\infty$. □

The properties (p) and (q) of the previous proposition are the natural dualizations of the properties used to characterize selectivity of coideals on $\mathbb{N}$ (see [23, Lemma 7.4]).

Using games, Halbeisen defines Ramsey families on $\mathbb{N}$ and also their dualizations, called Ramsey partition-families on $\mathcal{E}_\infty$. Every Ramsey family is a selective coideal on $\mathbb{N}$ (see [13, Fact 11.18]). The proof of this fact can be adapted to show that every Ramsey partition-family is a selective coideal on $\mathcal{E}_\infty$.

4.2. **Semiselective coideals.** We define now semiselective coideals of $\mathcal{E}_\infty$. This is a weakening of the notion of selectivity that will be used to develop the local Ramsey theory of the space $\mathcal{E}_\infty$.

Definition 4.9. Let $\mathcal{H} \subseteq \mathcal{E}_\infty$ be a coideal and $\mathcal{S} \subseteq \mathcal{H}$. A subset $\mathcal{D} \subseteq \mathcal{H}$ is *dense open* in $\mathcal{S}$ if $\mathcal{D} \subseteq \mathcal{S}$ and

(1) $(\forall M \in \mathcal{S})\,(\exists N \in \mathcal{D})\,(N \leq M)$,
(2) $(\forall M \in \mathcal{S})\,(\forall N \in \mathcal{D})\,(M \leq N \to M \in \mathcal{D})$.

Definition 4.10. Let $\mathcal{H} \subseteq \mathcal{E}_\infty$ be a coideal. Given $A \in \mathcal{H}$ and a collection $\mathcal{D} = \{\mathcal{D}_a\}_{a\in\mathcal{AE}_\infty \restriction A}$ such that each $\mathcal{D}_a$ is dense open in $\mathcal{H} \cap [\operatorname{depth}_A(a), A]$, we say that $B \leq A$ is a *diagonalization* of $\mathcal{D}$ if there exists a family $\mathcal{A} = \{A_a\}_{a\in\mathcal{AE}_\infty \restriction A} \subseteq \mathcal{H} \restriction A$, with each $A_a \in \mathcal{D}_a$, such that for every $a \in \mathcal{AE}_\infty \restriction B$, we have $[a, B] \subseteq [a, A_a]$.

Definition 4.11. We say that a coideal $\mathcal{H}$ on $\mathcal{E}_\infty$ is *semiselective* if for every $A \in \mathcal{H}$, every collection $\mathcal{D} = \{\mathcal{D}_a\}_{a\in\mathcal{AE}_\infty \restriction A}$ such that each $\mathcal{D}_a$ is dense open in $\mathcal{H} \cap [\operatorname{depth}_A(a), A]$, and every $B \in \mathcal{H} \restriction A$, there exists $C \in \mathcal{H} \restriction B$ such that C is a diagonalization of $\mathcal{D}$.

We show now that selectivity implies semiselectivity.

Lemma 4.12. *Given a coideal $\mathcal{H}$ on $\mathcal{E}_\infty$ and $A \in \mathcal{H}$, for every $\{\mathcal{D}_a\}_{a\in\mathcal{AE}_\infty \restriction A}$ such that each $\mathcal{D}_a$ is dense open in $\mathcal{H} \cap [\operatorname{depth}_A(a), A]$ there exists $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H} \restriction A$ such that $A_n \in \mathcal{D}_a$ for all $a \in \mathcal{AE}_\infty \restriction A$ with $\operatorname{depth}_A(a) = n$.*

Proof. For every $n \in \mathbb{N}$, list $\{a \in \mathcal{AE}_\infty : a \leq_{\text{fin}} r_n(A)\}$ as $\mathcal{A}_n = \{a^n_1, a^n_2, \dots, a^n_{k_n}\}$. Since each $\mathcal{D}_a$ is dense open in $\mathcal{H} \cap [\text{depth}_A(a), A]$, using A.3 mod $\mathcal{H}$ we can choose $A^{1,1} \in \mathcal{D}_{a^1_1} \restriction A$, $A^{1,2} \in \mathcal{D}_{a^1_2} \restriction A^{1,1}$, $\dots$, $A^{1,k_1} \in \mathcal{D}_{a^1_{k_1}} \restriction A^{1,k_1-1}$. Again, we can choose $A^{2,1} \in \mathcal{D}_{a^2_1} \restriction A^{1,k_1}$, $A^{2,2} \in \mathcal{D}_{a^2_2} \restriction A^{2,1}$, $\dots$, $A^{2,k_2} \in \mathcal{D}_{a^2_{k_2}} \restriction A^{2,k_2-1}$. And so on. For every $n \in \mathbb{N}$, let $A_n = A^{n,k_n}$. Then $\{A_n\}_{n\in\mathbb{N}}$ is as required. □

Theorem 4.13. *If $\mathcal{H} \subseteq \mathcal{E}_\infty$ is a selective coideal then $\mathcal{H}$ is semiselective.*

Proof. Consider $A \in \mathcal{H}$ and let $\mathcal{D} = \{\mathcal{D}_a\}_{a\in\mathcal{AE}_\infty\restriction A}$ be such that each $\mathcal{D}_a$ is dense open in $\mathcal{H} \cap [\text{depth}_A(a), A]$. Fix $B \in \mathcal{H} \restriction A$. Then $\mathcal{D}_a$ is dense open in $\mathcal{H} \cap [\text{depth}_A(a), B]$, for all $a \in \mathcal{AE}_\infty \restriction B$. Using Lemma 4.12 we can build $\mathcal{A} = \{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H} \restriction B$ such that $A_n \in \mathcal{D}_a$ for every $a \in \mathcal{AE}_\infty \restriction B$ with $\text{depth}_A(a) = n$. By selectivity, there exists $C \in \mathcal{H} \cap [\emptyset, B]$ which diagonalizes $\mathcal{A}$. Thus for every $b \in \mathcal{AE}_\infty \restriction C$ with $\text{depth}_A(b) = n$ we have $[b, C] \subseteq [b, A_n] \subseteq [b, A_b]$. Hence, $\mathcal{H}$ is semiselective. □

5. Semiselectivity and the Ramsey property

We will prove in this section that the families of $\mathcal{H}$-Ramsey sets and $\mathcal{H}$-Baire sets coincide if $\mathcal{H}$ is semiselective on $\mathcal{E}_\infty$ (see Theorem 5.6 below).

The following *combinatorial forcing* will be used:

Definition 5.1. Fix $\mathcal{F} \subseteq \mathcal{AE}_\infty$ and let $\mathcal{H} \subseteq \mathcal{E}_\infty$ be a coideal. Given $A \in \mathcal{H}$ and $a \in \mathcal{AE}_\infty$, we say that A *accepts* a if for every $B \in [a, A]$ there exists $n \in \mathbb{N}$ such that $r_n(B) \in \mathcal{F}$; we say that A *rejects* a if there is no $B \in [a, A] \cap \mathcal{H}$ such that B accepts a; and we say that A *decides* a if A either accepts or rejects a.

Lemma 5.2. *The combinatorial forcing has the following properties:*

(1) *If $A \in \mathcal{H}$ accepts a then every $B \in \mathcal{H} \restriction A$ with $[a, B] \neq \emptyset$ accepts a.*
(2) *If $A \in \mathcal{H}$ rejects a, then every $B \in \mathcal{H} \restriction A$ with $[a, B] \neq \emptyset$ rejects a.*
(3) *For every $A \in \mathcal{H}$ and every $a \in \mathcal{AE}_\infty \restriction A$ there exists $B \in [a, A] \cap \mathcal{H}$ which decides a.*
(4) *If $A \in \mathcal{H}$ accepts a then A accepts every $b \in r_{|a|+1}[a, A]$.*
(5) *If $A \in \mathcal{H}$ rejects a then there exists $B \in [a, A] \cap \mathcal{H}$ such that A does not accept any $b \in r_{|a|+1}[a, B]$.*

Proof. (1)–(4) follow easily from the definitions. Let us prove (5). Suppose that $A \in \mathcal{H}$ rejects a. Let $\mathcal{O} = \{b \in \mathcal{AE}_{|a|+1} : A \text{ accepts } b\}$. By A.4 mod $\mathcal{H}$, there exists $B \in \mathcal{H} \cap [a, A]$ such that $r_{|a|+1}[a, B] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, B] \subseteq \mathcal{O}^c$. If the first alternative holds, then take $C \in \mathcal{H} \cap [a, B]$ and let $b = r_{|a|+1}(C)$. Then $b \in \mathcal{O}$ and therefore A accepts b. Since $C \in [b, A]$, there exists n such that $r_n(C) \in \mathcal{F}$. Therefore B accepts a, because C is arbitrary. But this contradicts that A rejects a. Hence, $r_{|a|+1}[a, B] \subseteq \mathcal{O}^c$ and we are done. □

Lemma 5.3. *Let $\mathcal{H} \subseteq \mathcal{E}_\infty$ be a semiselective coideal. Given $A \in \mathcal{H}$ and $a \in \mathcal{AE}_\infty \restriction A$, there exists $D \in [a, A] \cap \mathcal{H}$ that decides every $b \in \mathcal{AE}_\infty \restriction D$ with $a \sqsubseteq b$.*

Proof. We prove the case $a = \emptyset$. The general case follows from a simple modification of the argument. For every $b \in \mathcal{AE}_\infty \restriction A$ define

$$\mathcal{D}_b = \{C \in \mathcal{H} \cap [\mathrm{depth}_A(b), A] : C \text{ decides } b\}.$$

By (1), (2) and (3) of Lemma 5.2, each $\mathcal{D}_b$ is dense open in $\mathcal{H} \cap [\mathrm{depth}_A(b), A]$. By semiselectivity, there exists $D \in \mathcal{H} \restriction A$ which diagonalizes the collection $\{\mathcal{D}_b\}_{b \in \mathcal{AE}_\infty \restriction A}$. By (1) and (2) of Lemma 5.2, D decides every $b \in \mathcal{AE}_\infty \restriction D$. □

Now, we adapt the semiselective Galvin lemma (see [8, 7]) to the space of infinite partitions of $\mathbb{N}$.

Lemma 5.4. *Given $\mathcal{F} \subseteq \mathcal{AE}_\infty$ and a semiselective coideal $\mathcal{H} \subseteq \mathcal{E}_\infty$, for every $A \in \mathcal{H}$ and every $a \in \mathcal{AE}_\infty \restriction A$, there exists $B \in \mathcal{H} \cap [a, A]$ such that one of the following holds:*

(1) $\mathcal{AE}_\infty \restriction [a, B] \cap \mathcal{F} = \emptyset$, *or*
(2) $(\forall C \in [a, B])\, (\exists\, n \in \mathbb{N})\, (r_n(C) \in \mathcal{F})$.

Proof. Consider $D \in [a, A] \cap \mathcal{H}$ as in Lemma 5.3. If D accepts a, condition (2) holds and we are done. So assume that D rejects a, and for $b \in \mathcal{AE}_\infty \restriction D$ such that $a \sqsubseteq b$, define

$$\mathcal{D}_b = \{C \in \mathcal{H} \cap [\mathrm{depth}_A(b), D] : C \text{ rejects every } b' \in r_{|b|+1}[b, C]\}$$

if D rejects b, and $\mathcal{D}_b = \mathcal{H} \cap [\mathrm{depth}_A(b), D]$, otherwise. By (2) and (5) of Lemma 5.2 each $\mathcal{D}_b$ is dense open in $\mathcal{H} \cap [\mathrm{depth}_A(b), D]$. By semiselectivity, choose $B \in \mathcal{H} \restriction D$ such that for all $b \in \mathcal{AE}_\infty \restriction B$ there exists $C_b \in \mathcal{D}_b$ with $[b, B] \subseteq [b, C_b]$. By induction on the length of b, with $a \sqsubseteq b$, and using that B is a diagonalization of the collection $\{\mathcal{D}_b\}_{b \in \mathcal{AE}_\infty \restriction D}$, it can be shown that B rejects every $b \in \mathcal{AE}_\infty \restriction B$ with $a \sqsubseteq b$. In fact, if D rejects b, then C_b rejects all $b' \in r_{|b|+1}[b, C_b]$. Hence, B satisfies that $\mathcal{AE}_\infty \restriction [a, B] \cap \mathcal{F} = \emptyset$. This completes the proof. □

Now, we give an application of Lemma 5.4. Recall that the basic metric open subsets of $\mathcal{E}_\infty$ are of the form $[b] = \{A \in \mathcal{E}_\infty : b \sqsubset A\}$, where $b \sqsubset A$ means that $b = r_n(A)$ for some $n \in \mathbb{N}$.

Theorem 5.5. *Suppose that $\mathcal{H} \subseteq \mathcal{E}_\infty$ is a semiselective coideal. Then the metric open subsets of $\mathcal{E}_\infty$ are $\mathcal{H}$-Ramsey.*

Proof. Let $\mathcal{X}$ be a metric open subset of $\mathcal{E}_\infty$ and fix a nonempty $[a, A]$ with $A \in \mathcal{H}$. Without loss of generality, we can assume $a = \emptyset$. Since $\mathcal{X}$ is open, there exists $\mathcal{F} \subseteq \mathcal{AE}_\infty$ such that $\mathcal{X} = \bigcup_{b \in \mathcal{F}} [b]$. Let $B \in \mathcal{H} \restriction A$ be as in Lemma 5.4. If (1) holds then $[\emptyset, B] \subseteq \mathcal{X}^c$ and if (2) holds then $[\emptyset, B] \subseteq \mathcal{X}$. □

The next theorem is the dual version of the semiselective Ellentuck theorem of [7] (see also [23]). It illustrates the strength of semiselective coideals of the space $\mathcal{E}_\infty$.

Theorem 5.6. *Let $\mathcal{H} \subseteq \mathcal{E}_\infty$ be a semiselective coideal and let $\mathcal{X} \subseteq \mathcal{E}_\infty$. Then, $\mathcal{X}$ is $\mathcal{H}$-Ramsey if and only if $\mathcal{X}$ is $\mathcal{H}$-Baire.*

Proof. Let $\mathcal{X}$ be an $\mathcal{H}$-Baire subset of $\mathcal{E}_\infty$. We will show that $\mathcal{X}$ is $\mathcal{H}$-Ramsey. Fix $A \in \mathcal{H}$. We prove the result for $[\emptyset, A]$ without loss of generality, that is, we will show that there is $A' \leq A$ such that $[\emptyset, A']$ is contained in $\mathcal{X}$ or is disjoint from $\mathcal{X}$.

For $a \in \mathcal{AE}_\infty \restriction A$ define

$$\mathcal{D}_a = \{B \in [\operatorname{depth}_A(a), A] \cap \mathcal{H} : [a, B] \subseteq \mathcal{X} \text{ or } [a, B] \subseteq \mathcal{X}^c \\ \text{or } [(\forall C \in [a, B])\,([a, C] \cap \mathcal{X} \neq \emptyset \text{ and } [a, C] \cap \mathcal{X}^c \neq \emptyset)]\}.$$

It is easy to see that each $\mathcal{D}_a$ is dense open in $\mathcal{H} \cap [\operatorname{depth}_A(a), A]$. By semiselectivity, choose $B \in \mathcal{H} \restriction A$ which diagonalizes the collection $\{\mathcal{D}_a\}_{a \in \mathcal{AE}_\infty \restriction A}$. Let $\mathcal{F}_0 = \{a \in \mathcal{AE}_\infty \restriction B : [a, B] \subseteq \mathcal{X}\}$ and $\mathcal{F}_1 = \{a \in \mathcal{AE}_\infty \restriction B : [a, B] \subseteq \mathcal{X}^c\}$. Consider $B_0 \in \mathcal{H} \restriction B$ as in Lemma 5.4 applied to $\mathcal{F}_0$ and B. If (2) of Lemma 5.4 holds then $[\emptyset, B_0] \subseteq \mathcal{X}$ and we are done. So assume that (1) holds. That is, $(\mathcal{AE}_\infty \restriction B_0) \cap \mathcal{F}_0 = \emptyset$. Now consider B_1 as in Lemma 5.4 applied to $\mathcal{F}_1$ and B_0. Again, if (2) holds then $[\emptyset, B_1] \subseteq \mathcal{X}^c$ and we are done. Notice that $(\mathcal{AE}_\infty \restriction B_1) \cap \mathcal{F}_1 \neq \emptyset$ because $(\mathcal{AE}_\infty \restriction B_1) \cap \mathcal{F}_0 = \emptyset$ and $\mathcal{X}$ is $\mathcal{H}$-Baire. So (2) holds. This concludes the proof. $\square$

From the previous proof it follows that if $\mathcal{H} \subseteq \mathcal{E}_\infty$ is a semiselective coideal and $\mathcal{X} \subseteq \mathcal{E}_\infty$, then $\mathcal{X}$ is $\mathcal{H}$-Ramsey null if and only if $\mathcal{X}$ is $\mathcal{H}$-nowhere dense.

Lemma 5.7. *If $\mathcal{H} \subseteq \mathcal{E}_\infty$ is a semiselective coideal, then the $\mathcal{H}$-Ramsey null sets form a σ-ideal.*

Proof. Let $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{H}$-Ramsey null subsets of $\mathcal{E}_\infty$, and let $[a, A]$ be a nonempty basic subset of $\mathcal{E}_\infty$ with $A \in \mathcal{H}$. We continue the proof assuming that $a = \emptyset$, since the general case is proved the same way. For every $b \in \mathcal{AE}_\infty \restriction A$, let

$$\mathcal{D}_b = \{B \in [\operatorname{depth}_A(b), A] \cap \mathcal{H} : (\forall n \leq |b|)\,([b, B] \cap \mathcal{X}_n = \emptyset)\}.$$

Each $\mathcal{D}_n$ is dense open in $[\operatorname{depth}_A(b)] \cap \mathcal{H}$, and so there is $C \in \mathcal{H} \restriction A$ and a collection $\{A_b\}_{b \in \mathcal{AE}_\infty \restriction A}$ such that $A_b \in \mathcal{D}_b$ for every b and $[b, C] \subseteq [b, A_b]$ for every $b \in \mathcal{AE}_\infty \restriction C$. Therefore, $[\emptyset, C] \cap \mathcal{X}_n = \emptyset$ for every $n \in \mathbb{N}$. Thus, $\bigcup_{n\in\mathbb{N}} \mathcal{X}_n$ is $\mathcal{H}$-Ramsey null. $\square$

It can also be shown that if $\mathcal{H} \subseteq \mathcal{E}_\infty$ is a semiselective coideal, the $\mathcal{H}$-Ramsey sets form a σ-algebra of subsets of $\mathcal{E}_\infty$.

We will now show that semiselectivity is in some sense the optimal property that a coideal must satisfy in order to get the equivalence in the statement of Theorem 5.6. First we give two characterizations of semiselectivity.

Proposition 5.8. *A coideal $\mathcal{H}$ on $\mathcal{E}_\infty$ is semiselective if and only if for every $A \in \mathcal{H}$ and every sequence $\mathcal{D} = \{\mathcal{D}_n\}_{n\in\mathbb{N}}$ with each $\mathcal{D}_n$ dense open in $[n, A] \cap \mathcal{H}$, for all $B \in \mathcal{H} \restriction A$ there is $C \in \mathcal{H} \restriction B$ and a sequence $\{A_n\}_{n\in\mathbb{N}}$ with $A_n \in \mathcal{D}_n$ for every n, such that $[a, C] \subseteq [a, A_n]$ for every $a \in \mathcal{AE}_\infty \restriction C$ with* $\operatorname{depth}_A(a) = n$.

Proof. Let $\mathcal{H}$ be a semiselective coideal, $A \in \mathcal{H}$, and $\mathcal{D} = \{\mathcal{D}_n\}_{n\in\mathbb{N}}$ with each $\mathcal{D}_n$ dense open in $[n, A] \cap \mathcal{H}$. Define $\mathcal{D}_a = \mathcal{D}_n$ if $\operatorname{depth}_A(a) = n$. Since $\mathcal{H}$ is semiselective, for every $B \in \mathcal{H}\restriction A$ there is $C \in \mathcal{H}\restriction B$ and $\{A_a\}_{a\in\mathcal{AE}_\infty\restriction A}$ with each $A_a \in \mathcal{D}_a = \mathcal{D}_n$ if $\operatorname{depth}_A(a) = n$, such that $[a, C] \subseteq [a, A_a]$ for every $a \in \mathcal{AE}_\infty\restriction C$.

For every n, put $A_n = \prod\{A_a : a \in \mathcal{AE}_\infty \restriction C \text{ and } \operatorname{depth}_A(a) = n\}$. Notice that $A_n \in \mathcal{E}_\infty$ since $\{a \in \mathcal{AE}_\infty : \operatorname{depth}_A(a) = n\}$ is finite and $C \leq^* A_a$ for every $a \in \mathcal{AE}_\infty \restriction C$. Then $A_n \in \mathcal{D}_n$ since $\mathcal{D}_n$ is dense open in $[n, A] \cap \mathcal{H}$. Since for every $a \in \mathcal{AE}_\infty\restriction C$ with $\operatorname{depth}_A(a) = n$, $[a, C] \subseteq [a, A_a]$, we have that for every partition X in $[a, C]$, each class of X whose minimum is greater than $p_n(X)$ is a union of classes of A_a, and this holds for each a with $\operatorname{depth}_A(a) = n$. This implies that each of those classes is a union of classes of A_n, and therefore $[a, C] \subseteq [a, A_n]$.

Conversely, let $A \in \mathcal{H}$ and $\{\mathcal{D}_a\}_{a\in\mathcal{AE}_\infty\restriction A}$ with each $\mathcal{D}_a$ dense open in $\mathcal{H} \cap [\operatorname{depth}_A(a), A]$. For every $n \in \mathbb{N}$ define $\mathcal{D}_n = \bigcap_{\operatorname{depth}_A(a)=n} \mathcal{D}_a$, so $\mathcal{D}_n$ is dense open in $\mathcal{H} \cap [n, A]$ since $\{a \in \mathcal{AE}_\infty : \operatorname{depth}_A(a) = n\}$ is finite. Then, for every $B \in \mathcal{H}\restriction A$ there is $C \in \mathcal{H} \restriction B$ and a sequence $\{A_n\}_{n\in\mathbb{N}}$, with each $A_n \in \mathcal{D}_n$, such that for every $a \in \mathcal{AE}_\infty \restriction C$ with $\operatorname{depth}_A(a) = n$, we have $[a, C] \subseteq [a, A_n]$, and $A_n \in \mathcal{D}_a$. Thus $\mathcal{H}$ is a semiselective coideal. □

We will now use two properties of a coideal $\mathcal{H} \subseteq \mathcal{E}_\infty$ to characterize semiselectivity. These are the dualizations of the corresponding properties of semiselective coideals on $\mathbb{N}$ (see [23, Lemma 7.9]).

Proposition 5.9. *A coideal $\mathcal{H}$ on $\mathcal{E}_\infty$ is semiselective if and only if it has the following two properties:*

- (wp) *For every $A \in \mathcal{H}$ and every sequence $\mathcal{D} = \{\mathcal{D}_n\}_{n\in\mathbb{N}}$ with each $\mathcal{D}_n$ dense open in $[n, A] \cap \mathcal{H}$, there is $B \in \mathcal{H} \restriction A$ and a sequence $\{A_n\}_{n\in\mathbb{N}}$ with $A_n \in \mathcal{D}_n$ for every n, such that $B \leq^* A_n$ for every n.*
- (q) *Given $X \in \mathcal{H}$ and an infinite partition of $p(X)$ into finite pieces, there is $Y \in \mathcal{H}\restriction X$ such that $p(Y)$ has at most one element in each piece.*

Proof. Let $\mathcal{H}$ be a semiselective coideal. Then by Fact 1.1 and Proposition 5.8, $\mathcal{H}$ has property (wp). We now prove it has property (q). Let $X \in \mathcal{H}$ and let $p(X) = \bigcup_{n\in\mathbb{N}} F_n$ be an infinite partition of $p(X)$ into finite pieces. For each n, let

$$\mathcal{D}_n = \{Y \in [n, X] \cap \mathcal{H} : |p(Y) \cap F_i| \leq 1 \text{ for every } i \leq n\}.$$

Then $\mathcal{D}_n$ is dense open in $[n, X] \cap \mathcal{H}$. Since $\mathcal{H}$ is a semiselective coideal, by Proposition 5.8 there is $C \in \mathcal{H}\restriction X$ and a sequence $\{A_n\}_{n\in\mathbb{N}}$ with $A_n \in \mathcal{D}_n$ for every n, such that for every $a \in \mathcal{AE}_\infty\restriction C$ with $\operatorname{depth}_A(a) = n$, $[a, C] \subseteq [a, A_n]$. Then $p(C)$ has at most one element in each piece F_n.

Conversely, suppose that $\mathcal{H}$ has properties (wp) and (q). Let $A \in \mathcal{H}$ and let $\{\mathcal{D}_n\}_{n\in\mathbb{N}}$ be a sequence such that for each n, $\mathcal{D}_n$ is dense open in $[n, A] \cap \mathcal{H}$. Then, by (wp) there is $B \in \mathcal{H}\restriction A$ and a sequence $\{A_n\}_{n\in\mathbb{N}}$ with $A_n \in \mathcal{D}_n$ for every n, such that $B \leq^* A_n$ for every n. So, for every n there is $i_n \in \mathbb{N}$ such that $B/i_n \leq A_n$; in other words, every class $[p_m(B)]$ of B with $m \geq i_n$ is a union of classes of A_n.

Define a strictly increasing sequence $\{m_i\}_{i\in\mathbb{N}}$ by $m_0 = 0$ and m_{k+1} the least $m > m_k$ such that $i_n \leq m$ for every $n \leq m_k$.

Consider now the infinite partition

$$p(B) = \bigcup_{k\in\mathbb{N}} \{p_i(B) : i \in [m_k, m_{k+1})\}.$$

By property (q), there is $C \in \mathcal{H}\restriction B$ such that for every k,

$$|\, p(C) \cap \{p_i(B) : i \in [m_k, m_{k+1})\}| \le 1.$$

Let $\mathcal{O} \subseteq \mathcal{AE}_1 \restriction C$ be the collection of 1-approximations of C with domain in an even numbered interval $[p_{m_{2k}}(B), p_{m_{2k+1}}(B))$. So,

$$\mathcal{O} = \{p_n(C) : n > 0 \text{ and } (\exists k)\ p_n(C) \in [p_{m_{2k}}(B), p_{m_{2k+1}}(B))\}.$$

By A.4 mod $\mathcal{H}$ there is $D \in \mathcal{H} \restriction C$ such that $p_n(D) \in \mathcal{O}$ for every $n > 0$ or $p_n(D) \notin \mathcal{O}$ for every $n > 0$. Then for every $a \in \mathcal{AE}_\infty \restriction D$, with $\operatorname{depth}_A(a) = n$, $[a, D] \subseteq [a, A_n]$. Thus, by Proposition 5.8, $\mathcal{H}$ is a semiselective coideal on $\mathcal{E}_\infty$. □

Proposition 5.10. *If $\mathcal{H} \subseteq \mathcal{E}_\infty$ is a coideal that fails to satisfy property* (q)*, then there is an $\mathcal{H}$-Baire subset of $\mathcal{E}_\infty$ that is not $\mathcal{H}$-Ramsey.*

Proof. If $\mathcal{H}$ does not satisfy property (q), there is $X \in \mathcal{H}$ and an infinite partition $p(X) = \bigcup_{n\in\mathbb{N}} F_n$ into finite pieces for which there is no $Y \in \mathcal{H} \restriction X$ such that $|p(Y) \cap F_n| \le 1$ for every n. Let

$$\mathcal{B} = \{Y \in \mathcal{E}_\infty : (\exists n)\ |p(Y) \cap F_n| \ge 2\}.$$

Notice that $\mathcal{B}$ is a metrically open subset of $\mathcal{E}_\infty$, and therefore it is $\mathcal{H}$-Baire. But $\mathcal{B}$ is not $\mathcal{H}$-Ramsey, since for every $Y \in \mathcal{H}\restriction X$ we have that $[\emptyset, Y] \cap \mathcal{B} \neq \emptyset$ and $[\emptyset, Y] \not\subseteq \mathcal{B}$. □

Proposition 5.11. *If $\mathcal{H} \subseteq \mathcal{E}_\infty$ is a coideal that fails to satisfy property* (wp)*, then the ideal of $\mathcal{H}$-Ramsey null sets is not a σ-ideal.*

Proof. Fix a sequence $\{\mathcal{D}_n\}_{n\in\mathbb{N}}$ and $A \in \mathcal{H}$ witnessing the failure of (wp). Then there is no $B \in \mathcal{H}\restriction A$ such that for all n there is $A_n \in \mathcal{D}_n$ with $B \le^* A_n$.

For every n let

$$\mathcal{X}_n = \{Y \in [n, A] : (\forall X \in \mathcal{D}_n)\,(Y \not\le^* X)\}.$$

Each $\mathcal{X}_n$ is $\mathcal{H}$-Ramsey null, because $\mathcal{D}_n$ is dense open in $[n, A] \cap \mathcal{H}$. But $\bigcup_{n\in\mathbb{N}} \mathcal{X}_n$ is not $\mathcal{H}$-Ramsey null, since for every $B \in [\emptyset, A]$ there is some n such that for all $X \in \mathcal{D}_n$, $B \not\le^* X$. □

To conclude this section we present a notion of Ramsey coideal of $\mathcal{E}_\infty$ and its relation to the concept of semiselectivity.

Definition 5.12. A coideal $\mathcal{H}$ on $\mathcal{E}_\infty$ is *Ramsey* if for every $A \in \mathcal{H}$ and every partition $f\colon \mathcal{AE}_2 \restriction A \to \{0, 1\}$, there exists $B \in \mathcal{H}\restriction A$ such that f is constant on $\mathcal{AE}_2 \restriction B$.

The following is a local version of Theorem 1.6 from [21], which in turn is an abstract version of Ramsey's Theorem from [22]:

Theorem 5.13. *Every semiselective coideal $\mathcal{H} \subseteq \mathcal{E}_\infty$ is Ramsey.*

Proof. Let $A \in \mathcal{H}$ and $f \colon \mathcal{AE}_2 \to \{0,1\}$ be given. For $a \in \mathcal{AE}_1 \upharpoonright A$, define

$$\mathcal{D}_a = \{B \in [\mathrm{depth}_A(a), A] \cap \mathcal{H} : f \text{ is constant on } r_2[a,B]\};$$

for all other cases, put $\mathcal{D}_a = \mathcal{H} \cap [\mathrm{depth}_A(a), A]$.

Using A.4 mod $\mathcal{H}$ in the case $a \in \mathcal{AE}_1 \upharpoonright A$, it is easy to prove that each $\mathcal{D}_a$ is dense open in $\mathcal{H} \cap [\mathrm{depth}_A(a), A]$. By semiselectivity, there exists $B_1 \in \mathcal{H} \upharpoonright A$ which diagonalizes the collection $\{\mathcal{D}_a\}_{a \in \mathcal{AE}_\infty \upharpoonright A}$. Notice that for every $a \in \mathcal{AE}_1 \upharpoonright B_1$, there exists $i_a \in \{0,1\}$ such that f takes constant value i_a on $r_2[a, B_1]$. Now, consider the partition $g \colon \mathcal{AE}_1 \upharpoonright B_1 \to \{0,1\}$ defined by $g(a) = i_a$. By A.4 mod $\mathcal{H}$ there exists $B \in \mathcal{H} \cap [\emptyset, B_1]$ such that g is constant on $r_1[\emptyset, B] = \mathcal{AE}_1 \upharpoonright B$. But $B \leq B_1 \leq A$, so $B \in \mathcal{H} \upharpoonright A$ as required. $\square$

6. Semiselectivity and forcing

In [10], Halbeisen presents a detailed study of forcing notions related to the space of partitions of $\mathbb{N}$. In this section we will consider the localization of these forcing notions to coideals and study properties of the forcing notions thus obtained.

For this section, some familiarity with the main ideas of forcing will be convenient.

6.1. **Forcing with $(\mathcal{H}, \leq^*)$.** Given a coideal $\mathcal{H} \subseteq \mathcal{E}_\infty$ we will consider the pair $(\mathcal{H}, \leq^*)$, where $\leq^*$ is the quasi-order "almost coarser" defined in the introduction.

Recall that for $A, B \in \mathcal{E}_\infty$, $A \leq^* B$ if there exists $a \in \mathcal{AE}_\infty \upharpoonright A$ such that $[a, A] \subseteq [a, B]$, according to Fact 1.1.

We view $(\mathcal{H}, \leq^*)$, for a semiselective coideal $\mathcal{H}$, as a forcing notion.

It will be convenient to consider also coideals that are maximal filters.

Definition 6.1. Given a nonempty family of partitions $\mathcal{U} \subseteq \mathcal{E}_\infty$, we say that $\mathcal{U}$ is an *ultrafilter* if it satisfies the following:

(a) $\mathcal{U}$ is a *filter* on $(\mathcal{E}_\infty, \leq)$, that is:
  (a.1) For all $A, B \in \mathcal{E}_\infty$, if $A \in \mathcal{U}$ and $A \leq B$ then $B \in \mathcal{U}$.
  (a.2) For all $A, B \in \mathcal{U}$, there exists $C \in \mathcal{U}$ such that $C \leq A$ and $C \leq B$.
(b) If $\mathcal{U}' \subseteq \mathcal{E}_\infty$ is a filter on $(\mathcal{E}_\infty, \leq)$ and $\mathcal{U} \subseteq \mathcal{U}'$ then $\mathcal{U}' = \mathcal{U}$. That is, $\mathcal{U}$ is a *maximal filter* on $(\mathcal{E}_\infty, \leq)$.
(c) (A.3 mod $\mathcal{U}$) For all $A \in \mathcal{U}$ and $a \in \mathcal{AE}_\infty \upharpoonright A$ with $\mathrm{depth}_A(a) = n$, the following holds:
  (c.1) $(\forall B \in [n, A] \cap \mathcal{U})\,([a, B] \cap \mathcal{U} \neq \emptyset)$.
  (c.2) $(\forall B \in [a, A] \cap \mathcal{U})\,(\exists A' \in [n, A] \cap \mathcal{U})\,([a, A'] \subseteq [a, B])$.
(d) (A.4 mod $\mathcal{U}$) Let $A \in \mathcal{U}$ and $a \in \mathcal{AE}_\infty \upharpoonright A$ be given. For all $\mathcal{O} \subseteq \mathcal{AE}_{|a|+1}$ there exists $B \in [\mathrm{depth}_A(a), A] \cap \mathcal{U}$ such that $r_{|a|+1}[a, B] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, B] \cap \mathcal{O} = \emptyset$.

Clearly, every ultrafilter over $\mathcal{E}_\infty$ is a coideal over $\mathcal{E}_\infty$, and is a filter over $\mathcal{E}_\infty$ in the sense of Halbeisen and Matet's Definition 4.4.

Lemma 6.2. *If $\mathcal{H} \subseteq \mathcal{E}_\infty$ is a semiselective coideal then $(\mathcal{H}, \leq^*)$ is σ-distributive.*

Proof. Let $\{\mathcal{D}_n\}_{n\in\mathbb{N}}$ be a countable collection of dense open subsets of $(\mathcal{H}, \leq^*)$. Fix $A \in \mathcal{H}$. For all $a \in \mathcal{AE}_\infty \restriction A$, the set $\mathcal{D}_a = \{B \in \mathcal{H} \cap [\mathrm{depth}_A(a), A] : B \in \mathcal{D}_{|a|}\}$ is dense open in $\mathcal{H} \cap [\mathrm{depth}_A(a), A]$. To show this, fix $a \in \mathcal{AE}_\infty \restriction A$. Obviously, if $B \in \mathcal{D}_a$ and $B' \in \mathcal{H} \cap [\mathrm{depth}_A(a), A]$ is such that $B' \leq B$ then $B' \in \mathcal{D}_a$. Now, given $C \in \mathcal{H} \cap [\mathrm{depth}_A(a), A]$, choose $B_a \in \mathcal{D}_{|a|}$ such that $B_a \leq^* C$. Then there exists $b \in \mathcal{AE}_\infty \restriction B_a$ such that $[b, B_a] \subseteq [b, C]$. We will assume that $\mathrm{depth}_C(b) \geq \mathrm{depth}_C(a) = \mathrm{depth}_A(a)$ (otherwise, let $m = |b| + \mathrm{depth}_C(a)$, choose $D \in [b, B_a]$ and let $\hat{b} = r_{m+1}(D)$; then, $[\hat{b}, B_a] \subseteq [\hat{b}, C]$ and $\mathrm{depth}_C(\hat{b}) \geq \mathrm{depth}_C(a)$). By A.3 mod $\mathcal{H}$, choose $B \in \mathcal{H} \cap [\mathrm{depth}_C(b), C]$ such that $\emptyset \neq [b, B] \subseteq [b, B_a]$. So $B \leq^* B_a$ and therefore $B \in \mathcal{D}_{|a|}$. Notice also that since $\mathrm{depth}_C(b) \geq \mathrm{depth}_C(a) = \mathrm{depth}_A(a)$, we have that $B \in \mathcal{H} \cap [\mathrm{depth}_A(a), C] \subseteq \mathcal{H} \cap [\mathrm{depth}_A(a), A]$. This implies that $B \leq C$ and $B \in \mathcal{D}_a$, completing the proof that $\mathcal{D}_a$ is dense open. Let $B \in \mathcal{H} \restriction A$ be a diagonalization of $\{\mathcal{D}_a\}_{a\in\mathcal{AE}_\infty\restriction A}$. Then there exists a family $\{A_a\}_{a\in\mathcal{AE}_\infty\restriction A}$ with $A_a \in \mathcal{D}_a$ such that $[a, B] \subseteq [a, A_a]$ for all $a \in \mathcal{AE}_\infty \restriction B$. This implies that $B \leq^* A_a$ for all $a \in \mathcal{AE}_\infty \restriction B$. Therefore, $B \in \mathcal{D}_{|a|}$ for all $a \in \mathcal{AE}_\infty \restriction B$. That is, $B \in \bigcap_n \mathcal{D}_n$, and the proof is complete. □

Lemma 6.3. *Let $\mathcal{H} \subseteq \mathcal{E}_\infty$ be a semiselective coideal. Forcing with $(\mathcal{H}, \leq^*)$ adds no new elements of $\mathcal{AE}_\infty^{\mathbb{N}}$ (in particular, no new elements of $\mathcal{E}_\infty$ or $\mathcal{H}$), and if $\mathcal{U}$ is the $(\mathcal{H}, \leq^*)$-generic filter over some ground model V, then $\mathcal{U}$ is a selective ultrafilter in $V[\mathcal{U}]$.*

Proof. Each element of $\mathcal{E}_\infty$ can be coded by a subset of $\mathbb{N}$, and also every sequence in $\mathcal{AE}_\infty^{\mathbb{N}}$ can be coded by a subset of $\mathbb{N}$. Therefore, since $(\mathcal{H}, \leq^*)$ is σ-distributive, the fact that forcing with $(\mathcal{H}, \leq^*)$ adds no new elements of $\mathcal{AE}_\infty^{\mathbb{N}}$ follows by a standard argument (see for instance [15, Theorem 15.6]). Let $\mathcal{U}$ be the $(\mathcal{H}, \leq^*)$-generic filter over some ground model V. By genericity, $\mathcal{U}$ is a maximal filter. Also by genericity, A.3 and A.4 (for the space $\mathcal{E}_\infty$), we have that A.3 mod $\mathcal{U}$ and A.4 mod $\mathcal{U}$ hold (and therefore, $\mathcal{U}$ satisfies Definition 6.1). Also, since $\mathcal{E}_\infty$ is closed, given $A \in \mathcal{U}$, the set of diagonalizations of any sequence $\mathcal{A} = (A_n)_{n\in\mathbb{N}} \subseteq \mathcal{U} \restriction A$ is dense in $(\mathcal{H}, \leq^*)$. Therefore, by genericity, $\mathcal{U}$ is selective. □

Corollary 6.4. *Let $\mathcal{U}$ be the $(\mathcal{H}, \leq^*)$-generic filter over some ground model V. Then $\mathcal{U}$ is Ramsey in $V[\mathcal{U}]$.*

Proof. It turns out that the generic ultrafilter $\mathcal{U}$ is a selective coideal. Then, by Theorems 5.13 and 4.13, $\mathcal{U}$ is Ramsey. □

Lemma 6.5. *Suppose $\mathcal{H}$ is not semiselective. Let $\mathcal{U}$ be a $(\mathcal{H}, \leq^*)$-generic filter over some ground model V. Then $\mathcal{U}$ is not selective in $V[\mathcal{U}]$.*

Proof. Since $\mathcal{H}$ is not semiselective, there exist $A \in \mathcal{H}$ and a collection $\mathcal{D} = \{\mathcal{D}_a\}_{a\in\mathcal{AE}_\infty\restriction A}$ with $\mathcal{D}_a$ dense open in $\mathcal{H} \cap [\mathrm{depth}_A(a), A]$, for every $a \in \mathcal{AE}_\infty \restriction A$, and $\mathcal{D}$ has no diagonalization in $\mathcal{H}$. In $V[\mathcal{U}]$, each $\mathcal{D}_a$ is dense in $(\mathcal{H} \restriction A, \leq^*)$. Proceeding as in Lemma 4.12, find a decreasing sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H} \restriction A$ such that $A_n \in \mathcal{U} \cap \mathcal{D}_a$, for every $a \in \mathcal{AE}_\infty \restriction A$ with $\mathrm{depth}_A(a) = n$. Then $\{A_n\}_{n\in\mathbb{N}}$

has no diagonalization in $\mathcal{U}$ and therefore $\mathcal{U}$ is not selective. This completes the proof. $\square$

The following theorem summarizes all the results of this section.

Theorem 6.6. *Let $\mathcal{H} \subseteq \mathcal{E}_\infty$ be a coideal. The following are equivalent:*

(1) *$\mathcal{H}$ is semiselective.*
(2) *Forcing with $(\mathcal{H}, \leq^*)$ adds no new elements of $\mathcal{AE}_\infty^{\mathbb{N}}$ (in particular, no new elements of $\mathcal{E}_\infty$ or $\mathcal{H}$), and if $\mathcal{U}$ is the $(\mathcal{H}, \leq^*)$-generic filter over some ground model V, then $\mathcal{U}$ is a selective ultrafilter in $V[\mathcal{U}]$.*

6.2. **Forcing with $\mathbb{M}_\mathcal{H}$.** Let $\mathcal{H}$ be a semiselective coideal. In this section we will study the forcing notion for $\mathcal{E}_\infty$ which is a dualization of Mathias forcing introduced in [20]. The partial order is defined by:

$$\mathbb{M}_\mathcal{H} = \{(a, A) : A \in \mathcal{H},\, a \in \mathcal{AE}_\infty \restriction A\} \cup \{\emptyset\},$$

with $(a, A) \leq (b, B)$ if and only if $[a, A] \subseteq [b, B]$. We say that $\mathbb{M}_\mathcal{H}$ is the *Mathias forcing notion* associated to $\mathcal{H}$. As is customary in the forcing terminology, the elements of $\mathbb{M}_\mathcal{H}$ are sometimes called conditions.

This forcing notion is the dual Mathias forcing but with conditions that have their second coordinate in the family $\mathcal{H}$. We will prove that if $\mathcal{H}$ is a semiselective coideal, this forcing has properties similar to those of the dual Mathias forcing presented in [11] and [13].

A partition $X \in \mathcal{E}_\infty$ is $\mathbb{M}_\mathcal{H}$-*generic* over a model V if for every dense open subset $\mathcal{D}$ of $\mathbb{M}_\mathcal{H}$, such that $\mathcal{D} \in V$, there exists a condition $(a, A) \in \mathcal{D}$ such that $X \in [a, A]$.

The coideal $\mathcal{H}$ has the *pure decision property* (or the Prikry property) if for every sentence of the forcing language ϕ and every condition $(a, A) \in \mathbb{M}_\mathcal{H}$ there exists $B \in [a, A] \cap \mathcal{H}$ such that (a, B) decides ϕ. The coideal $\mathcal{H}$ has the *hereditary genericity property* (or the Mathias property) if it satisfies that if X is $\mathbb{M}_\mathcal{H}$-generic over a model V, then every $Y \leq X$ is $\mathbb{M}_\mathcal{H}$-generic over V.

Theorem 6.7. *If $\mathcal{H} \subseteq \mathcal{E}_\infty$ is a semiselective coideal then it has the pure decision property.*

Proof. Suppose that $\mathcal{H}$ is semiselective and fix a sentence ϕ of the forcing language, and a condition $(a, A) \in \mathbb{M}_\mathcal{H}$. For every $b \in \mathcal{AE}_\infty \restriction A$ with $a \sqsubseteq b$, let $\mathcal{D}_b = \{B \in \mathcal{H} \cap [\mathrm{depth}_A(b), A] : (b, B)$ decides ϕ or $(\forall C \in \mathcal{H} \cap [b, B])\,(b, C)$ does not decide $\phi\}$, and for $a \not\sqsubseteq b$, set $\mathcal{D}_b = \mathcal{H} \cap [\mathrm{depth}_A(b), A]$, for all $b \in \mathcal{AE}_\infty \restriction A$.

Each $\mathcal{D}_b$ is dense open in $\mathcal{H} \cap [\mathrm{depth}_A(b), A]$. Fix a diagonalization $B \in \mathcal{H} \restriction A$. Let

$$\mathcal{F}_0 = \{b \in \mathcal{AE}_\infty \restriction B : a \sqsubseteq b \;\&\; (b, B) \text{ forces } \phi\},$$
$$\mathcal{F}_1 = \{b \in \mathcal{AE}_\infty \restriction B : a \sqsubseteq b \;\&\; (b, B) \text{ forces } \neg\phi\}.$$

Let $\hat{C} \in \mathcal{H} \restriction B$ as in Lemma 5.4 applied to B and $\mathcal{F}_0$, and let $C \in \mathcal{H} \restriction \hat{C}$ be as in Lemma 5.4 applied to $\hat{C}$ and $\mathcal{F}_1$. Let us prove that (a, C) decides ϕ. Let (b_0, C_0) and (b_1, C_1) be two different extensions of (a, C); suppose that (b_0, C_0) forces ϕ

and (b_1, C_1) forces $\neg\phi$. Then $b_0 \in \mathcal{F}_0$ and $b_1 \in \mathcal{F}_1$. But $b_0, b_1 \in \mathcal{AE}_\infty \restriction C$, so by the choice of C this means that every element of $\mathcal{H} \cap [a, C]$ has an initial segment in $\mathcal{F}_0$ and an initial segment in $\mathcal{F}_1$. So there exist two compatible extensions of (a, C) such that one forces ϕ and the other forces $\neg\phi$ — a contradiction. Thus either both (b_0, C_0) and (b_1, C_1) force ϕ or both (b_0, C_0) and (b_1, C_1) force $\neg\phi$. Therefore (a, C) decides ϕ. □

When $\mathcal{H}$ is the whole space $\mathcal{E}_\infty$, this is the pure decision property of the dual-Mathias forcing ([13, Theorem 28.2]).

Now, we will prove that if $\mathcal{H} \subseteq \mathcal{E}_\infty$ is semiselective then it has the hereditary genericity property (Theorem 6.13 below). Given a selective ultrafilter $\mathcal{U} \subset \mathcal{E}_\infty$, let $\mathbb{M}_\mathcal{U}$ be the set of all pairs (a, A) such that $A \in \mathcal{U}$ and $[a, A] \neq \emptyset$. Order $\mathbb{M}_\mathcal{U}$ with the same ordering used for $\mathbb{M}_\mathcal{H}$.

For the rest of this section we essentially follow [21], where the case in which $\mathcal{H}$ is the whole topological Ramsey space is treated. We include the proofs here for completeness.

Definition 6.8. Let $\mathcal{U} \subseteq \mathcal{E}_\infty$ be a selective ultrafilter, $\mathcal{D}$ a dense open subset of $\mathbb{M}_\mathcal{U}$, and $a \in \mathcal{AE}_\infty$. We say that A *captures* $(a, \mathcal{D})$ if $A \in \mathcal{U}$, $[a, A] \neq \emptyset$, and for all $B \in [a, A]$ there exists $m > |a|$ such that $(r_m(B), A) \in \mathcal{D}$.

Lemma 6.9. *Let $\mathcal{U} \subseteq \mathcal{E}_\infty$ be a selective ultrafilter and $\mathcal{D}$ a dense open subset of $\mathbb{M}_\mathcal{U}$. Then, for every $a \in \mathcal{AE}_\infty$ there exists A which captures $(a, \mathcal{D})$.*

Proof. Given $a \in \mathcal{AE}_\infty$, choose $B \in \mathcal{U}$ such that $[a, B] \neq \emptyset$. We can define a collection $\{C_b\}_{b \in \mathcal{AE}_\infty \restriction B}$ with $[b, C_b] \neq \emptyset$ such that:

(1) For all $b_1, b_2 \in \mathcal{AE}_\infty \restriction B$, if $\operatorname{depth}_B(b_1) = \operatorname{depth}_B(b_2)$ then $C_{b_1} = C_{b_2}$.
(2) For all $b_1, b_2 \in \mathcal{AE}_\infty \restriction B$, if $b_1 \sqsubseteq b_2$ then $C_{b_1} \geq C_{b_2}$.
(3) For all $b \in \mathcal{AE}_\infty \restriction B$ with $a \sqsubseteq b$, either $(b, C_b) \in \mathcal{D}$ or if such a $C_b \in \mathcal{D}$ does not exist then $C_b = B$.

For every $b \in \mathcal{AE}_\infty \restriction B$, let $C_n = C_b$ if $\operatorname{depth}_B(b) = n$. Notice that $C_n \geq C_{n+1}$, for all every $n \in \mathbb{N}$. Let $C \in \mathcal{U} \cap [a, B]$ be a diagonalization of $\{C_n\}_{n \in \mathbb{N}}$. Then, for all $b \in \mathcal{AE}_\infty \restriction C$ with $a \sqsubseteq b$, if there exists a $\hat{C} \in \mathcal{U}$ such that $(b, \hat{C}) \in \mathcal{D}$, we must have $(b, C) \in \mathcal{D}$.

Let $\mathcal{X} = \{D \in \mathcal{E}_\infty : D \leq C \to (\exists b \in \mathcal{AE}_\infty \restriction D)\ a \sqsubset b\ \&\ (b, C) \in \mathcal{D}\}$. $\mathcal{X}$ is a metric open subset of $\mathcal{E}_\infty$ and therefore, by Theorem 5.5, it is $\mathcal{U}$-Ramsey. Take $\hat{C} \in \mathcal{U} \cap [\operatorname{depth}_C(a), C]$ such that $[a, \hat{C}] \subseteq \mathcal{X}$ or $[a, \hat{C}] \cap \mathcal{X} = \emptyset$. We will show that the first alternative holds: Pick $A \in \mathcal{U} \cap [a, \hat{C}]$ and $(a', A') \in \mathcal{D}$ such that $(a', A') \leq (a, A)$. Notice that $a \sqsubseteq a'$ and therefore, by (3), we have $(a', C) \in \mathcal{D}$. By the definition of $\mathcal{X}$, we also have $A' \in \mathcal{X}$. Now choose $A'' \in \mathcal{U} \cap [a', A']$. Then (a', A'') is also in $\mathcal{D}$ and therefore $A'' \in \mathcal{X}$. But $A'' \in [a', A'] \subseteq [a, A] \subseteq [a, \hat{C}]$. This implies that $[a, \hat{C}] \subseteq \mathcal{X}$. Finally, that A captures $(a, \mathcal{D})$ follows from the definition of $\mathcal{X}$ and the fact that $[a, A] \subseteq [a, \hat{C}] \subseteq [a, C]$. This completes the proof. □

Theorem 6.10. *Let $\mathcal{U} \subseteq \mathcal{E}_\infty$ be a selective ultrafilter in a given transitive model V of $ZF + DCR$. Forcing over V with $\mathbb{M}_\mathcal{U}$ adds a generic $g \in \mathcal{E}_\infty$ with the property that $g \leq^* A$ for all $A \in \mathcal{U}$. In fact, $B \in \mathcal{E}_\infty$ is $\mathbb{M}_\mathcal{U}$-generic over V if and only if $B \leq^* A$ for all $A \in \mathcal{U}$.*

Proof. Suppose that $B \in \mathcal{E}_\infty$ is $\mathbb{M}_\mathcal{U}$-generic over V. Fix an arbitrary $A \in \mathcal{U}$. The set $\{(c, C) \in \mathbb{M}_\mathcal{U} : C \leq^* A\}$ is dense open and is in V. Fix $(a, A') \in \mathbb{M}_\mathcal{U}$. Choose $C_0 \in \mathcal{U}$ such that $C_0 \leq A, A'$. Since $\mathcal{U}$ is an ultrafilter, we can choose $C_1 \in \mathcal{U}$ and $n \in \mathbb{N}$ such that $[n, C_1] \subseteq [a, A'] \cap [1, C_0]$. Let $c = r_n(C_1)$. By A.3 mod $\mathcal{U}$, there exists $C_2 \in \mathcal{U} \cap [\mathrm{depth}_{A'}(c), A']$ such that $\emptyset \neq [c, C_2] \subseteq [c, C_1]$. It is clear that $[c, C_2] \subseteq [c, A]$ and therefore $C_2 \leq^* A$. Also, since $\mathrm{depth}_{A'}(c) \geq \mathrm{depth}_{A'}(a)$, we have $[a, C_2] \neq \emptyset$. Thus, $(a, C_2) \leq (a, A')$. That is, $\mathcal{D}$ is dense. It is obviously open. So, by genericity, there exists one $(c, C) \in \mathcal{D}$ such that $B \in [c, C]$. Hence $B \leq^* A$.

Now, suppose that $B \in \mathcal{E}_\infty$ is such that $B \leq^* A$ for all $A \in \mathcal{U}$, and let $\mathcal{D}$ be a dense open subset of $\mathbb{M}_\mathcal{U}$. We need to find $(a, A) \in \mathcal{D}$ such that $B \in [a, A]$. In V, by using Lemma 6.9 iteratively, we can define a sequence $\{A_n\}_{n\in\mathbb{N}}$ such that $A_n \in \mathcal{U}$, $A_{n+1} \leq A_n$ and A_n captures $(r_n(B), \mathcal{D})$. Since $\mathcal{U}$ is a selective ultrafilter in V, we can choose $A \in \mathcal{U}$, in V, such that $A \leq^* A_n$ for all n. By our assumption on B, we have $B \leq^* A$. So there exists an $a \in \mathcal{AE}_\infty$ such that $[a, B] \subseteq [a, A]$. Let $m = \mathrm{depth}_B(a)$. By A.3 mod $\mathcal{U}$, we can assume that $a = r_m(B) = r_m(A)$, and also that $A \in [r_m(B), A_m]$. Therefore, $B \in [m, A]$ and A captures $(r_m(B), \mathcal{D})$. Hence, the following is true in V:

$$(\forall C \in [m, A])\, (\exists n > m)\, ((r_n(C), A) \in \mathcal{D}). \tag{6.1}$$

Let $\mathcal{F} = \{b : (\exists n > m)\, (b \in r_n[m, A]\ \&\ (b, A) \notin \mathcal{D})\}$ and give $\mathcal{F}$ the strict end-extension ordering $\sqsubset$. Then the relation $(\mathcal{F}, \sqsubset)$ is in V, and by equation (6.1) $(\mathcal{F}, \sqsubset)$ is well-founded. Therefore, by a well-known argument due to Mostowski, equation (6.1) holds in the universe. Hence, since $B \in [m, A]$, there exists $n > m$ such that $(r_n(B), A) \in \mathcal{D}$. But $B \in [r_n(B), A]$, so B is $\mathbb{M}_\mathcal{U}$-generic over V. $\square$

Corollary 6.11. *If B is $\mathbb{M}_\mathcal{U}$-generic over some model V and $A \leq B$ then A is also $\mathbb{M}_\mathcal{U}$-generic over V.*

Lemma 6.12. *Let $\mathcal{H} \subseteq \mathcal{E}_\infty$ be a semiselective coideal. Consider the forcing notion $\mathbb{P} = (\mathcal{H}, \leq^*)$ and let $\hat{\mathcal{U}}$ be a $\mathbb{P}$-name for a $\mathbb{P}$-generic ultrafilter. Then the iteration $\mathbb{P} * \mathbb{M}_{\hat{\mathcal{U}}}$ is equivalent to the forcing $\mathbb{M}_\mathcal{H}$.*

Proof. Recall that $\mathbb{P} * \mathbb{M}_{\hat{\mathcal{U}}} = \{(B, (\dot{a}, \dot{A})) : B \in \mathcal{H}\ \&\ B \vdash (\dot{a}, \dot{A}) \in \mathbb{M}_{\hat{\mathcal{U}}}\}$, with the ordering $(B, (\dot{a}, \dot{A})) \leq (B_0, (\dot{a}_0, \dot{A}_0)) \Leftrightarrow B \leq^* B_0\ \&\ (\dot{a}, \dot{A}) \leq (B_0, (\dot{a}_0, \dot{A}_0))$. The mapping $(a, A) \mapsto (A, (\hat{a}, \hat{A}))$ is a dense embedding from $\mathbb{M}_\mathcal{H}$ to $\mathbb{P} * \mathbb{M}_{\hat{\mathcal{U}}}$ (here $\hat{a}$ and $\hat{A}$ are the canonical $\mathbb{P}$-names for a and A, respectively). It is easy to show that this mapping preserves the order. So, given $(B, (\dot{a}, \dot{A})) \in \mathbb{P} * \mathbb{M}_{\hat{\mathcal{U}}}$, we need to find $(d, D) \in \mathbb{M}_\mathcal{H}$ such that $(D, (\hat{d}, \hat{D})) \leq (B, (\dot{a}, \dot{A}))$. Since $\mathbb{P}$ is σ-distributive, there exist $a \in \mathcal{AE}_\infty$, $A \in \mathcal{H}$ and $C \leq *B$ in $\mathcal{H}$ such that $C \Vdash_\mathbb{P} (\hat{a} = \dot{a}\ \&\ \hat{A} = \dot{A})$ (so we can assume that $a \in \mathcal{AE}_\infty \restriction C$). Notice that $(C, (\hat{a}, \hat{A})) \in \mathbb{P} * \mathbb{M}_{\hat{\mathcal{U}}}$ and $(C, (\hat{a}, \hat{A})) \leq (B, (\dot{a}, \dot{A}))$. So, $C \Vdash_\mathbb{P} \hat{C} \in \hat{\mathcal{U}}$ and $C \Vdash_\mathbb{P} \hat{A} \in \hat{\mathcal{U}}$. Then, $C \Vdash_\mathbb{P} (\exists x \in \hat{\mathcal{U}})\, (x \in [\hat{a}, \hat{A}]\ \&\ x \in [\hat{a}, \hat{C}])$.

So there exists $D \in \mathcal{H}$ such that $D \in [a, A] \cap [a, C]$. Hence, $(D, (\hat{a}, \hat{D})) \leq (B, (\dot{a}, \dot{A}))$. This completes the proof. $\square$

The next theorem follows immediately from Corollary 6.11 and Lemma 6.12.

Theorem 6.13. *Every semiselective coideal $\mathcal{H} \subseteq \mathcal{E}_\infty$ has the hereditary genericity property.*

The forcing notion $\mathbb{M}_\mathcal{H}$ is the dualization of Mathias forcing to the space $\mathcal{E}_\infty$. In [20] Mathias used this forcing to prove that all definable collections of infinite subsets of $\mathbb{N}$ are $\mathcal{H}$-Ramsey in Solovay's model, obtained by the collapse of a Mahlo cardinal to $\aleph_1$, where $\mathcal{H}$ is a selective coideal on $\mathbb{N}$ which belongs to that model. For this, the properties of pure decision and hereditary genericity are important. Similar results can be proved for the space $\mathcal{E}_\infty$, but this is out of the scope of this article.

References

[1] T. J. Carlson and S. G. Simpson, A dual form of Ramsey's theorem, *Adv. in Math.* **53** (1984), no. 3, 265–290. MR 0753869.

[2] T. J. Carlson and S. G. Simpson, Topological Ramsey theory, in *Mathematics of Ramsey Theory*, 172–183, Algorithms Combin., 5, Springer, Berlin, 1990. MR 1083600.

[3] J. Cichoń, A. Krawczyk, B. Majcher-Iwanow and B. Węglorz, Dualization of the van Douwen diagram, *J. Symbolic Logic* **65** (2000), no. 2, 959–968. MR 1771096.

[4] C. Di Prisco, J. G. Mijares and J. Nieto, Local Ramsey theory: an abstract approach, *MLQ Math. Log. Q.* **63** (2017), no. 5, 384–396. MR 3748482.

[5] C. Di Prisco, J. G. Mijares and C. Uzcátegui, Ideal games and Ramsey sets, *Proc. Amer. Math. Soc.* **140** (2012), no. 7, 2255–2265. MR 2898689.

[6] E. Ellentuck, A new proof that analytic sets are Ramsey, *J. Symbolic Logic* **39** (1974), 163–165. MR 0349393.

[7] I. Farah, Semiselective coideals, *Mathematika* **45** (1998), no. 1, 79–103. MR 1644345.

[8] F. Galvin, A generalization of Ramsey's theorem, *Notices Amer. Math. Soc.* **15** (1968), 548. https://www.ams.org/journals/notices/196804/196804FullIssue.pdf.

[9] F. Galvin and K. Prikry, Borel sets and Ramsey's theorem, *J. Symbolic Logic* **38** (1973), 193–198. MR 0337630.

[10] L. Halbeisen, Symmetries between two Ramsey properties, *Arch. Math. Logic* **37** (1998), no. 4, 241–260. MR 1635558.

[11] L. Halbeisen, On shattering, splitting and reaping partitions, *Math. Logic Quart.* **44** (1998), no. 1, 123–134. MR 1601919.

[12] L. Halbeisen, Ramseyan ultrafilters, *Fund. Math.* **169** (2001), no. 3, 233–248. MR 1852127.

[13] L. J. Halbeisen, *Combinatorial Set Theory: With a Gentle Introduction to Forcing*, Second Edition, Springer Monographs in Mathematics, Springer, Cham, 2017. MR 3751612.

[14] L. Halbeisen and P. Matet, A generalization of the dual Ellentuck theorem, *Arch. Math. Logic* **42** (2003), no. 2, 103–128. MR 1961873.

[15] T. Jech, *Set Theory*, the third millennium edition, revised and expanded, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. MR 1940513.

[16] I. G. Kastanas, On the Ramsey property for sets of reals, *J. Symbolic Logic* **48** (1983), no. 4, 1035–1045 (1984). MR 0727792.

[17] P. Matet, Partitions and filters, *J. Symbolic Logic* **51** (1986), no. 1, 12–21. MR 0830067.

[18] P. Matet, Some filters of partitions, *J. Symbolic Logic* **53** (1988), no. 2, 540–553. MR 0947858.

[19] P. Matet, Happy families and completely Ramsey sets, *Arch. Math. Logic* **32** (1993), no. 3, 151–171. MR 1201647.

[20] A. R. D. Mathias, Happy families, *Ann. Math. Logic* **12** (1977), no. 1, 59–111. MR 0491197.

[21] J. G. Mijares, A notion of selective ultrafilter corresponding to topological Ramsey spaces, *MLQ Math. Log. Q.* **53** (2007), no. 3, 255–267. MR 2330595.

[22] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* (2) **30** (1929), no. 4, 264–286. MR 1576401.

[23] S. Todorcevic, *Introduction to Ramsey Spaces*, Annals of Mathematics Studies, 174, Princeton University Press, Princeton, NJ, 2010. MR 2603812.

J. C. Cano
Universidad Nacional de Colombia
juccanora@unal.edu.co

C. A. Di Prisco ✉
Universidad de Los Andes
ca.di@uniandes.edu.co

Received: May 30, 2021
Accepted: September 16, 2021