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ON ESSENTIAL SELF-ADJOINTNESS OF SINGULAR STURM–LIOUVILLE OPERATORS

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Abstract. Considering singular Sturm–Liouville differential expressions of the type

$$\tau_\alpha = -(d/dx)x^\alpha(d/dx) + q(x), \quad x \in (0,b),\, \alpha \in \mathbb{R},$$

we employ some Sturm comparison-type results in the spirit of Kurss to derive criteria for τ_α to be in the limit-point and limit-circle case at $x = 0$. More precisely, if $\alpha \in \mathbb{R}$ and, for $0 < x$ sufficiently small,

$$q(x) \geq [(3/4) - (\alpha/2)]x^{\alpha-2},$$

or, if $\alpha \in (-\infty, 2)$ and there exist $N \in \mathbb{N}$ and $\varepsilon > 0$ such that, for $0 < x$ sufficiently small,

$$q(x) \geq [(3/4) - (\alpha/2)]x^{\alpha-2} - (1/2)(2-\alpha)x^{\alpha-2}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-1} + [(3/4)+\varepsilon]x^{\alpha-2}[\ln_1(x)]^{-2},$$

then τ_α is nonoscillatory and in the limit-point case at $x = 0$. Here iterated logarithms for $0 < x$ sufficiently small are of the form

$$\ln_1(x) = |\ln(x)| = \ln(1/x), \qquad \ln_{j+1}(x) = \ln(\ln_j(x)), \quad j \in \mathbb{N}.$$

Analogous results are derived for τ_α to be in the limit-circle case at $x = 0$. We also discuss a multi-dimensional application to partial differential expressions of the type

$$-\operatorname{div}|x|^\alpha\nabla + q(|x|), \quad \alpha \in \mathbb{R},\, x \in B_n(0;R)\backslash\{0\},$$

with $B_n(0;R)$ the open ball in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, centered at $x = 0$ of radius $R \in (0,\infty)$.

1. Introduction

In a nutshell, we are interested in deriving limit-point and limit-circle criteria for singular differential expressions of the type

$$\tau_\alpha = -(d/dx)x^\alpha(d/dx) + q(x), \quad x \in (0,b),\, \alpha \in \mathbb{R}. \tag{1.1}$$

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The principal results we prove in Theorems 3.1 and 3.3 on the basis of Sturm comparison-type results initiated by Kurss are the following: Suppose that $\alpha \in \mathbb{R}$ and, for $0 < x$ sufficiently small,

$$q(x) \geq [(3/4) - (\alpha/2)]x^{\alpha-2}, \tag{1.2}$$

or, in the context of logarithmic refinements of (1.2), assume that $\alpha \in (-\infty, 2)$ and that there exist $N \in \mathbb{N}$ and $\varepsilon > 0$ such that, for $0 < x$ sufficiently small,

$$\begin{aligned} q(x) \geq [(3/4) - (\alpha/2)]x^{\alpha-2} - (1/2)(2-\alpha)x^{\alpha-2} \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1} \\ + [(3/4) + \varepsilon]x^{\alpha-2}[\ln_1(x)]^{-2}. \end{aligned} \tag{1.3}$$

(For the definition of the iterated logarithms $\ln_\ell(\cdot)$, $\ell \in \mathbb{N}$, we refer the reader to (3.3).) Then, in either situation (1.2) and (1.3), τ_α is nonoscillatory and in the limit-point case at $x = 0$.

Similarly, if $\alpha \in (-\infty, 2)$ and there exists $\varepsilon \in (0, 1)$ (depending on α) such that, for $0 < x$ sufficiently small (depending on ε),

$$q(x) \leq [(3/4) - (\alpha/2) - \varepsilon]x^{\alpha-2}, \tag{1.4}$$

or, if $\alpha \in (-\infty, 2)$ and there exist $N \in \mathbb{N}$ and $\varepsilon \in (0, 1)$ (depending on α and N), such that, for $0 < x$ sufficiently small (depending on ε and N),

$$\begin{aligned} q(x) \leq [(3/4) - (\alpha/2)]x^{\alpha-2} - (1/2)(2-\alpha)x^{\alpha-2} \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1} \\ - [\varepsilon(2-\alpha)/2]x^{\alpha-2} \prod_{k=1}^{N} [\ln_k(x)]^{-1}. \end{aligned} \tag{1.5}$$

Then, in either situation (1.4) and (1.5), τ_α is in the limit-circle case at $x = 0$.

We note that the amount of literature on limit-point/limit-circle criteria for Sturm–Liouville operators is so immense that there is no possibility to account for it in this short paper. The reader can find very thorough discussions of this circle of ideas in [6, Ch. XIII.6, Sect. XIII.10.D], [10, Ch. 4.6], [18, Sect. 23.6], [21, App. to Sect. X.1], [30, Sect. 13.4], [31, Ch. 7], and the extensive literature cited therein. One of the driving motivations for writing this paper was to emphasize the simplicity of proofs of Theorems 3.1 and 3.3 (essentially, they are reduced to certain computations), given the Sturm comparison results, Theorems 2.8 and 2.9, in the spirit of Kurss.

We now briefly turn to the content of each section: Section 2 provides the necessary background on minimal and maximal operators associated with general three-coefficient Sturm–Liouville differential expressions, and it recalls Weyl's limit-point/limit-circle classification and some oscillation theory, as well as Sturm comparison results, quoting some extensions of the classical work by Kurss [17]. Section 3 contains our principal results, Theorems 3.1 and 3.3, as summarized in

(1.2)–(1.5). Section 4 considers an elementary multi-dimensional application to partial differential expressions of the type

$$-\operatorname{div}|x|^{\alpha}\nabla + q(|x|), \quad \alpha \in \mathbb{R},\ x \in B_n(0;R)\backslash\{0\},$$

with $B_n(0;R)$ the open ball in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, centered at $x=0$ of radius $R \in (0,\infty)$. Finally, Appendix A contains some more involved computations needed in the proofs of Theorems 3.1 and 3.3.

2. Some background on Sturm–Liouville operators

In this section we briefly recall the necessary background on maximal and minimal operators corresponding to three-coefficient Sturm–Liouville differential expressions, introduce the notion of deficiency indices for symmetric Sturm–Liouville operators, discuss Weyl's limit-point/limit-circle dichotomy, recall some oscillation theory, and discuss an extension of Sturm comparison results, applicable to the limit-point/limit-circle case, following Kurss [17]. Standard references for much of this material are, for instance, [3, Ch. 6], [4, Chs. 8, 9], [6, Sects. XIII.6, XIII.9, XIII.10], [10, Ch. 4], [11, Ch. III], [18, Ch. V], [19], [20, Ch. 6], [27, Ch. 9], [28, Sect. 8.3], [30, Ch. 13], [31, Chs. 4, 6–8].

We start with the basic set of assumptions throughout this section:

Hypothesis 2.1. *Let $(a,b) \subseteq \mathbb{R}$ and suppose that p, q, r are (Lebesgue) measurable functions on (a,b) such that the following three conditions hold:*

(i) *$r > 0$ a.e. on (a,b), $r \in L^1_{\text{loc}}((a,b);dx)$.*

(ii) *$p > 0$ a.e. on (a,b), $1/p \in L^1_{\text{loc}}((a,b);dx)$.*

(iii) *q is real-valued a.e. on (a,b), $q \in L^1_{\text{loc}}((a,b);dx)$.*

Given Hypothesis 2.1, we briefly study Sturm–Liouville differential expressions τ of the type,

$$\tau = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right] \quad \text{for a.e. } x \in (a,b) \subseteq \mathbb{R}. \tag{2.1}$$

Given τ as in (2.1), the corresponding *maximal operator T_{max}* in $L^2((a,b);r\,dx)$ associated with τ is defined by

$$\begin{aligned} &T_{max} f = \tau f, \\ &f \in \operatorname{dom}(T_{max}) = \big\{g \in L^2((a,b);r\,dx) \,\big|\, g, g^{[1]} \in AC_{\text{loc}}((a,b)); \\ &\qquad\qquad\qquad\qquad\qquad\qquad \tau g \in L^2((a,b);r\,dx)\big\}, \end{aligned} \tag{2.2}$$

with $g^{[1]} = pg'$ denoting the first quasi-derivative of g. The *preminimal operator $\dot{T}_{min}$* in $L^2((a,b);r\,dx)$ associated with τ is defined by

$$\begin{aligned} &\dot{T}_{min} f = \tau f, \\ &f \in \operatorname{dom}\big(\dot{T}_{min}\big) = \big\{g \in L^2((a,b);r\,dx) \,\big|\, g, g^{[1]} \in AC_{\text{loc}}((a,b)); \\ &\qquad\qquad \operatorname{supp}(g) \subset (a,b) \text{ is compact}; \tau g \in L^2((a,b);r\,dx)\big\}. \end{aligned} \tag{2.3}$$

One can prove that $\dot{T}_{min}$ is closable and then define the *minimal operator* T_{min} in $L^2((a,b); r\,dx)$ as the closure of $\dot{T}_{min}$,

$$T_{min} = \overline{\dot{T}_{min}}. \tag{2.4}$$

The following result recalls Weyl's celebrated alternative:

Theorem 2.2. *Assume Hypothesis* 2.1. *Then the following alternative holds:*

(i) *For every $z \in \mathbb{C}$, all solutions u of $(\tau - z)u = 0$ are in $L^2((a,b); r\,dx)$ near b (resp., near a).*

(ii) *For every $z \in \mathbb{C}$, there exists at least one solution u of $(\tau - z)u = 0$ which is not in $L^2((a,b); r\,dx)$ near b (resp., near a). In this case, for each $z \in \mathbb{C}\backslash\mathbb{R}$, there exists precisely one solution u_b (resp., u_a) of $(\tau - z)u = 0$ (up to constant multiples) which lies in $L^2((a,b); r\,dx)$ near b (resp., near a).*

This naturally leads to the notion that τ is in the limit-point or limit-circle case at an interval endpoint as follows:

Definition 2.3. Assume Hypothesis 2.1.
In case (i) in Theorem 2.2, τ is said to be in the *limit-circle case* at b (resp., at a).
In case (ii) in Theorem 2.2, τ is said to be in the *limit-point case* at b (resp., at a).

The deficiency indices of T_{min} are then given by

$$\begin{aligned} n_{\pm}(T_{min}) &= \dim(\ker(T_{max} \mp iI)) \\ &= \begin{cases} 2 & \text{if } \tau \text{ is in the limit-circle case at } a \text{ and } b, \\ 1 & \text{if } \tau \text{ is in the limit-circle case at } a \\ & \quad \text{and in the limit-point case at } b, \text{ or vice versa}, \\ 0 & \text{if } \tau \text{ is in the limit-point case at } a \text{ and } b. \end{cases} \end{aligned} \tag{2.5}$$

In particular, $T_{min} = T_{max}$ is self-adjoint $\big($i.e., $\dot{T}_{min}$ is essentially self-adjoint$\big)$ if and only if τ is in the limit-point case at a and b, underscoring the special role played by limit-point endpoints (as opposed to a limit-circle endpoint that requires a boundary condition in connection with self-adjointness issues of T_{min}).

We continue with a few remarks on Sturm's oscillation theory (see, e.g., [10, Theorem 7.4.4], [29, Sect. 14], and the detailed list of references cited therein).

Definition 2.4. Assume Hypothesis 2.1.

(i) Fix $c \in (a,b)$ and $\lambda \in \mathbb{R}$. Then $\tau - \lambda$ is called *nonoscillatory* at a (resp., at b), if there exists a real-valued solution $u(\lambda, \cdot)$ of $\tau u = \lambda u$ that has finitely many zeros in (a,c) (resp., in (c,b)). Otherwise, $\tau - \lambda$ is called *oscillatory* at a (resp., at b).

(ii) Let $\lambda_0 \in \mathbb{R}$. Then T_{min} is called *bounded from below* by λ_0, and one writes $T_{min} \geq \lambda_0 I$, if

$$(u, [T_{min} - \lambda_0 I]u)_{L^2((a,b); r\,dx)} \geq 0, \quad u \in \operatorname{dom}(T_{min}). \tag{2.6}$$

Remark 2.5. By Sturm's separation theorem, $\tau - \lambda$, $\lambda \in \mathbb{R}$, is nonoscillatory at a (resp., at b) if and only if every real-valued solution $u(\lambda, \cdot)$ of $\tau u = \lambda u$ has finitely many zeros in (a, c) (resp., in (c, b)).

The following is a key result relating the notions of boundedness from below and nonoscillation.

Theorem 2.6. *Assume Hypothesis* 2.1*. Then the following two conditions are equivalent:*

(i) T_{min} *(and hence any symmetric extension of* T_{min}*) is bounded from below.*
(ii) *There exists* $\nu_0 \in \mathbb{R}$ *such that, for all* $\lambda < \nu_0$*,* $\tau - \lambda$ *is nonoscillatory at a and b.*

We also recall Sturm's comparison result in the following form.

Theorem 2.7. *Assume that* p*,* q_j*,* r *satisfy Hypothesis* 2.1 *and set*

$$\tau_j = r(x)^{-1}[-(d/dx)p(x)(d/dx) + q_j(x)] \quad \text{for a.e. } x \in (a,b) \subseteq \mathbb{R},\ j = 1, 2. \tag{2.7}$$

Fix $\lambda \in \mathbb{R}$ *and let* u_j *be a real-valued solution of* $\tau_j u_j = \lambda u_j$*,* $j = 1, 2$*. If, for some* $x_0 \in (a, b) \subseteq \mathbb{R}$*,*

$$\begin{aligned} &q_2 \geq q_1 \quad \text{a.e. on } (a,b), \\ &u_1 \neq 0 \quad \text{on } (a,b)\backslash\{x_0\}, \\ &u_2(x_0) = u_1(x_0), \quad u_2^{[1]}(x_0) = u_1^{[1]}(x_0), \end{aligned} \tag{2.8}$$

then

$$|u_2(x)| \geq |u_1(x)| \quad \text{for all } x \in (a,b); \tag{2.9}$$

in particular,

$$u_2 \neq 0 \quad \text{on } (a,b)\backslash\{x_0\}. \tag{2.10}$$

Next, we also recall the following general comparison result (due to Kurss [17] in the special case[1] $r = 1$):

Theorem 2.8. *Assume that* p*,* q_j*,* r *satisfy Hypothesis* 2.1 *and set*

$$\tau_j = r(x)^{-1}[-(d/dx)p(x)(d/dx) + q_j(x)] \quad \text{for a.e. } x \in (a,b) \subseteq \mathbb{R},\ j = 1, 2. \tag{2.11}$$

Suppose that τ_1 *is nonoscillatory and in the limit-point case at a and that* $q_2 \geq q_1$ *a.e. on* $(a, b) \subseteq \mathbb{R}$*. Then* τ_2 *is also nonoscillatory and in the limit-point case at a. The analogous statement applies to the endpoint b.*

Theorem 2.8 is a consequence of Theorem 2.7 (cf. [10, Theorem 7.4.6]), as is its limit-circle analogue below:

Theorem 2.9. *Under the assumptions of Theorem* 2.8*, suppose* τ_1 *is nonoscillatory and in the limit-circle case at a, and that* $q_2 \leq q_1$ *a.e. on* $(a, b) \subseteq \mathbb{R}$*. Then* τ_2 *is in the limit-circle case at a. The analogous statement applies to the endpoint b.*

[1] One of us, F. G., is indebted to Hubert Kalf for kindly and repeatedly sharing his detailed notes on the proof of Theorem 2.8 and on various ramifications of this circle of ideas (private communications, May 2015 and September 2017).

Proof. By assumption, both fundamental solutions $u_{1,1}$ and $u_{1,2}$ of $\tau_1 u_1 = 0$ are strictly positive on (a, x_0) for some $x_0 \in (a,b)$ with $u_{1,j} \in L^2((a,x_0); r\,dx)$, $j \in \{1,2\}$. Let $u_{2,j}$, $j \in \{1,2\}$, be the solutions of $\tau_2 u_2 = 0$ with initial data $u_{2,j}(x_0) = u_{1,j}(x_0)$ and $u_{2,j}^{[1]}(x_0) = u_{1,j}^{[1]}(x_0)$. Then, by Theorem 2.7, $|u_{2,j}(x)| \leq |u_{1,j}(x)|$ for all $x \in (a, x_0)$, which implies $u_{2,j} \in L^2((a,x_0); r\,dx)$, $j \in \{1,2\}$. Thus τ_2 is in the limit-circle case at a. □

3. Results on the limit-point and limit-circle case

In this section we prove our principal limit-point/limit-circle results on the special two-coefficient differential expression τ_α in (3.2) below at $x = 0$.

Specializing to the case of half-line two-coefficient Sturm–Liouville operators, where

$$\begin{aligned} &a = 0, \quad b \in (0,\infty) \cup \{\infty\}, \quad p(x) = x^\alpha,\ \alpha \in \mathbb{R}, \quad r(x) = 1,\ x \in (0,b), \\ &q \in L^1_{\text{loc}}((0,b); dx) \text{ is real-valued a.e.,} \end{aligned} \tag{3.1}$$

τ in (2.1) now takes on the simplified form

$$\tau_\alpha = -(d/dx) x^\alpha (d/dx) + q(x) \quad \text{for a.e. } x \in (0,b),\ \alpha \in \mathbb{R}. \tag{3.2}$$

Given $\alpha \in \mathbb{R}$, or $\alpha \in (-\infty, 2)$, we will derive conditions on q that imply the limit-point (and nonoscillatory) or limit-circle behavior of τ_α at $x = 0$.

Introducing iterated logarithms for $0 < x$ sufficiently small,

$$\ln_1(x) = |\ln(x)| = \ln(1/x), \qquad \ln_{j+1}(x) = \ln(\ln_j(x)), \quad j \in \mathbb{N} \tag{3.3}$$

(rendering $\ln_\ell(\cdot)$, $1 \leq \ell \leq N$, in Theorems 3.1 and 3.3 below, strictly positive), our first principal result reads as follows:

Theorem 3.1. *Suppose that $q \in L^1_{\text{loc}}((0,b); dx)$ is real-valued a.e. on $(0,b)$.*

(i) *Let $\alpha \in \mathbb{R}$ and assume that, for a.e. $0 < x$ sufficiently small,*

$$q(x) \geq [(3/4) - (\alpha/2)] x^{\alpha - 2}. \tag{3.4}$$

Then τ_α is nonoscillatory and in the limit-point case at $x = 0$.

(ii) *Let[2] $\alpha \in (-\infty, 2)$ and assume there exist $N \in \mathbb{N}$ and $\varepsilon > 0$ such that, for a.e. $0 < x$ sufficiently small (depending on N and ε),*

$$\begin{aligned} q(x) \geq{} & [(3/4) - (\alpha/2)] x^{\alpha-2} - (1/2)(2-\alpha) x^{\alpha-2} \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1} \\ & + [(3/4) + \varepsilon] x^{\alpha-2} [\ln_1(x)]^{-2} \equiv Q_{\alpha,N,\varepsilon}(x). \end{aligned} \tag{3.5}$$

Then τ_α is nonoscillatory and in the limit-point case at $x = 0$.

Proof. In the following, $0 < x$ is assumed to be sufficiently small.

[2] Only $\alpha \in (-\infty, 2)$ can improve on item (i).

(i) Abbreviating, for $\alpha \in \mathbb{R}$,

$$q_{\alpha,0}(x) = [(3/4) - (\alpha/2)]x^{\alpha-2}, \tag{3.6}$$

$$y_0(x) = x^{-1/2}, \tag{3.7}$$

$$\tau_{\alpha,0} = -(d/dx)x^{\alpha}(d/dx) + q_{\alpha,0}(x), \tag{3.8}$$

one confirms that

$$(\tau_{\alpha,0}\, y_0)(x) = 0, \quad \alpha \in \mathbb{R}. \tag{3.9}$$

In particular, (3.7) and (3.9) prove that $\tau_{\alpha,0}$, $\alpha \in \mathbb{R}$, is nonoscillatory at $x = 0$. Moreover, since for $R > 0$,

$$y_0 \notin L^2((0,R);dx), \quad \alpha \in \mathbb{R}, \tag{3.10}$$

$\tau_{\alpha,0}$, $\alpha \in \mathbb{R}$, is nonoscillatory and in the limit-point case at $x = 0$, and hence so is τ_{α}, $\alpha \in \mathbb{R}$, by Theorem 2.8.

(ii) Since the sum of the second and third terms on the right-hand side of (3.5) would be nonnegative for $\alpha \geq 2$, Theorem 2.8 yields that only the case $\alpha \in (-\infty, 2)$ can improve upon item (i). Next, we abbreviate, for $\alpha \in (-\infty, 2)$, $N \in \mathbb{N}$, and $0 < x$ sufficiently small,[3]

$$\begin{aligned} q_{\alpha,N}(x) = {} & [(3/4) - (\alpha/2)]x^{\alpha-2} - (1/2)(2-\alpha)x^{\alpha-2}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_{\ell}(x)]^{-1} \\ & + (3/4)x^{\alpha-2}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_{\ell}(x)]^{-2} \\ & + x^{\alpha-2}\sum_{j=1}^{N-1}\prod_{\ell=1}^{j}[\ln_{\ell}(x)]^{-2}\sum_{m=j+1}^{N}\prod_{p=j+1}^{m}[\ln_{p}(x)]^{-1}, \end{aligned} \tag{3.11}$$

$$y_N(x) = x^{-1/2}\prod_{k=1}^{N}[\ln_k(x)]^{-1/2}, \tag{3.12}$$

$$\tau_{\alpha,N} = -(d/dx)x^{\alpha}(d/dx) + q_{\alpha,N}(x), \tag{3.13}$$

and claim (cf. Lemma A.1) that

$$(\tau_{\alpha,N}\, y_N)(x) = 0, \quad \alpha \in (-\infty, 2),\, N \in \mathbb{N}. \tag{3.14}$$

Once more, (3.12) and (3.14) prove that $\tau_{\alpha,N}$, $\alpha \in (-\infty, 2)$, $N \in \mathbb{N}$, is nonoscillatory at $x = 0$. Moreover, since for $0 < \delta_N$ sufficiently small,

$$y_N \notin L^2((0,\delta_N);dx), \tag{3.15}$$

[3] If $N = 1$ one interprets, as usually, sums and products over empty index sets as 0 and 1, respectively.

$\tau_{\alpha,N}$, $\alpha \in (-\infty, 2)$, $N \in \mathbb{N}$, is nonoscillatory and in the limit-point case at $x = 0$. Since, for $0 < \varepsilon$ and $0 < x$ both sufficiently small, one infers that

$$\begin{aligned} &(3/4)x^{\alpha-2}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-2} + x^{\alpha-2}\sum_{j=1}^{N-1}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-2}\sum_{m=j+1}^{N}\prod_{p=j+1}^{m}[\ln_p(x)]^{-1} \\ &\quad \leq [(3/4)+\varepsilon]x^{\alpha-2}[\ln_1(x)]^{-2}, \end{aligned} \tag{3.16}$$

condition (3.5) implies that $q(x) \geq Q_{\alpha,N,\varepsilon}(x) \geq q_{\alpha,N}(x)$ for $0 < \varepsilon$ and $0 < x$ sufficiently small. Thus, Theorem 2.8 implies that also τ_α, $\alpha \in (-\infty, 2)$, is nonoscillatory and in the limit-point case at $x = 0$. ☐

Although not needed in the context of Theorem 3.1, we note that a second (necessarily nonoscillatory) linearly independent solution $\widetilde{y}_N$ of $\tau_{\alpha,N} y = 0$ is a consequence of the standard reduction of order approach:

$$\widetilde{y}_N(x) = y_N(x)\int_x^c dt\, t^{-\alpha} y_N(t)^{-2}, \quad 0 < x < c \text{ sufficiently small.} \tag{3.17}$$

Remark 3.2. In connection with the nonoscillatory behavior of τ_α, we now recall the power-weighted and logarithmically refined Hardy inequalities in the form (see [7] and the references therein)

$$\begin{aligned} &\int_0^\rho dx\, x^\alpha |f'(x)|^2 \geq \frac{(1-\alpha)^2}{4}\int_0^\rho dx\, x^{\alpha-2}|f(x)|^2, \\ &\qquad \alpha \in \mathbb{R},\ \rho \in (0,\infty)\cup\{\infty\},\ f \in C_0^\infty((0,\rho)), \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \int_0^\rho dx\, x^\alpha |f'(x)|^2 &\geq \frac{(1-\alpha)^2}{4}\int_0^\rho dx\, x^{\alpha-2}|f(x)|^2 \\ &\quad + \frac{1}{4}\sum_{j=1}^{N}\int_0^\rho dx\, x^{\alpha-2}\Bigg(\prod_{\ell=1}^{j}[\ln_\ell(x/\gamma)]^{-2}\Bigg)|f(x)|^2, \\ &N \in \mathbb{N},\ \alpha \in \mathbb{R},\ \rho,\gamma \in (0,\infty),\ \gamma \geq e_N\rho,\ f \in C_0^\infty((0,\rho)), \end{aligned} \tag{3.19}$$

where

$$e_0 = 0, \qquad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}. \tag{3.20}$$

Inequalities (3.18) and (3.19) imply, in particular, that

$$\begin{aligned} &\bigg(-\frac{d}{dx}x^\alpha\frac{d}{dx} - \frac{(1-\alpha)^2}{4}x^{\alpha-2}\bigg)\bigg|_{C_0^\infty((0,\rho))} \geq 0, \\ &\qquad \alpha \in \mathbb{R},\ \rho \in (0,\infty)\cup\{\infty\}, \end{aligned} \tag{3.21}$$

and

$$\bigg(-\frac{d}{dx}x^{\alpha}\frac{d}{dx}-\frac{(1-\alpha)^2}{4}x^{\alpha-2}-\frac{1}{4}x^{\alpha-2}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_{\ell}(x/\gamma)]^{-2}\bigg)\bigg|_{C_0^{\infty}((0,\rho))}\geq 0,$$
$$N\in\mathbb{N},\ \alpha\in\mathbb{R},\ \rho,\gamma\in(0,\infty),\ \gamma\geq e_N\rho. \quad (3.22)$$

All constants displayed in (3.18)–(3.22) are sharp (cf. [7]).

Since $[(3/4)-(\alpha/2)]\geq -(1-\alpha)^2/4$ is equivalent to $(\alpha-2)^2\geq 0$, which automatically holds for all $\alpha\in\mathbb{R}$, the assertion that $\tau_{\alpha,0}$, $\alpha\in\mathbb{R}$, is nonoscillatory (cf. the paragraph following (3.9)) is of course consistent with Hardy's inequality (3.18), which implies the (much weaker) inequality

$$\bigg(-\frac{d}{dx}x^{\alpha}\frac{d}{dx}+\bigg(\frac{3}{4}-\frac{\alpha}{2}\bigg)x^{\alpha-2}\bigg)\bigg|_{C_0^{\infty}((0,\rho))}\geq 0,$$
$$\alpha\in\mathbb{R},\ \ \rho\in(0,\infty)\cup\{\infty\}. \quad (3.23)$$

Similarly, the assertion that $\tau_{\alpha,N}$, $\alpha\in\mathbb{R}$, $N\in\mathbb{N}$, is nonoscillatory (cf. the paragraph following (3.14)) is consistent with the logarithmic refinements of Hardy's inequality (3.19) as $[(3/4)-(\alpha/2)]>-(1-\alpha)^2/4$ is equivalent to $(\alpha-2)^2/4>0$, which in turn automatically holds for all $\alpha\in(-\infty,2)$. The subtle difference $[(3/4)-(\alpha/2)]>-(1-\alpha)^2/4$ versus $[(3/4)-(\alpha/2)]\geq-(1-\alpha)^2/4$ now is crucial as the leading logarithmic terms in (3.11) are of the form $[\ln_{\ell}(x)]^{-1}$ as opposed to the smaller terms $[\ln_{\ell}(x)]^{-2}$ in (3.19) and (3.22). In particular, (3.22) now implies

$$\bigg(-\frac{d}{dx}x^{\alpha}\frac{d}{dx}+\bigg(\frac{3}{4}-\frac{\alpha}{2}\bigg)x^{\alpha-2}+\frac{\alpha-2}{2}x^{\alpha-2}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_{\ell}(x/\gamma)]^{-1}\bigg)\bigg|_{C_0^{\infty}((0,\rho))}\geq 0,$$
$$N\in\mathbb{N},\ \alpha\in\mathbb{R},\ \gamma\geq e_N\rho, \quad (3.24)$$

for suitable $\rho,\gamma\in(0,\infty)$, since for arbitrarily small $0<\delta\equiv(\alpha-2)^2/4$, the expression $\delta x^{\alpha-2}$ dominates the second term on the right-hand side of (3.11) in a sufficiently small neighborhood of $x=0$.

Our second main result then reads as follows:

Theorem 3.3. *Suppose that $q\in L^1_{\text{loc}}((0,b);dx)$ is real-valued a.e. on $(0,b)$.*

(i) *Let $\alpha\in(-\infty,2)$ and assume that there exists $\varepsilon\in(0,1)$ (depending on α) such that, for a.e. $0<x$ sufficiently small (depending on ε),*

$$q(x)\leq[(3/4)-(\alpha/2)-\varepsilon]x^{\alpha-2}. \quad (3.25)$$

Then τ_{α} is in the limit-circle case at $x=0$.

(ii) *Let $\alpha \in (-\infty, 2)$ and assume there exist $N \in \mathbb{N}$ and $\varepsilon \in (0,1)$ (depending on α and N) such that, for a.e. $0 < x$ sufficiently small (depending on N and ε),*

$$\begin{aligned} q(x) \leq [(3/4) - (\alpha/2)]x^{\alpha-2} - (1/2)(2-\alpha)x^{\alpha-2} \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1} \\ - (\varepsilon/2)(2-\alpha)x^{\alpha-2} \prod_{k=1}^{N} [\ln_k(x)]^{-1} \equiv \widehat{Q}_{\alpha,N,\varepsilon}(x). \end{aligned} \tag{3.26}$$

Then, τ_α is in the limit-circle case at $x = 0$.

Proof. In the following, $0 < x$ is assumed to be sufficiently small.

(i) Abbreviating, for $\alpha \in (-\infty, 2)$, $\beta \in (0, \infty)$,

$$q_{\alpha,0,\beta}(x) = [(3/4) - (\alpha/2) - \beta]x^{\alpha-2}, \tag{3.27}$$

$$\begin{aligned} & y_{\alpha,0,\beta,j}(x) = x^{\gamma_{\alpha,\beta,j}}, \\ & \gamma_{\alpha,\beta,j} = (1/2)(1-\alpha) - (1/2)(-1)^j \begin{cases} \big|(2-\alpha)^2 - 4\beta\big|^{1/2}, & 0 < 4\beta \leq (2-\alpha)^2, \\ i\big|(2-\alpha)^2 - 4\beta\big|^{1/2}, & (2-\alpha)^2 \leq 4\beta, \end{cases} \\ & \qquad \alpha \in (-\infty, 2),\ \beta \in (0,\infty)\backslash\big\{(2-\alpha)^2/4\big\},\ j = 1, 2, \end{aligned} \tag{3.28}$$

$$\begin{aligned} & y_{\alpha,0,(2-\alpha)^2/4,1}(x) = x^{(1-\alpha)/2}, \quad y_{\alpha,0,(2-\alpha)^2/4,2}(x) = x^{(1-\alpha)/2}\ln(1/x), \\ & \qquad \gamma_{\alpha,(2-\alpha)^2/4,j} = (1-\alpha)/2,\ j = 1, 2, \end{aligned} \tag{3.29}$$

$$\tau_{\alpha,0,\beta} = -(d/dx)x^\alpha(d/dx) + q_{\alpha,0,\beta}(x), \tag{3.30}$$

one confirms that

$$(\tau_{\alpha,0,\beta}\, y_{\alpha,0,\beta,j})(x) = 0, \quad \alpha \in (-\infty, 2),\ \beta \in (0,\infty),\ j = 1, 2. \tag{3.31}$$

To verify the limit-circle property of $\tau_{\alpha,0,\beta}$, one needs to guarantee that, for some $\rho \in (0,1)$, $y_{\alpha,0,\beta,j} \in L^2((0,\rho); dx)$, $j = 1, 2$; equivalently,

$$\operatorname{Re}(\gamma_{\alpha,\beta,j}) > -1/2, \quad j = 1, 2. \tag{3.32}$$

Inequality (3.32) in turn is equivalent to

$$\alpha \in (-\infty, 2), \quad \beta \in (0, \infty). \tag{3.33}$$

Hence, choosing $\alpha \in (2, \infty)$ and $\beta \equiv \varepsilon \in \big(0, (2-\alpha)^2/4\big]$ yields that for $0 < \varepsilon$ sufficiently small, $\tau_{\alpha,0,\varepsilon}$ is in the limit-circle case and nonoscillatory (cf. (3.28)) at $x = 0$. An application of Theorem 2.9 then yields that also τ_α is in the limit-circle case at $x = 0$.

(ii) We recall that $q_{\alpha,N}$ is given by (3.11), and abbreviate, for $N \in \mathbb{N}$ and $0 < x$ sufficiently small,[4]

$$
\begin{aligned}
q_{\alpha,N,\varepsilon}(x) = q_{\alpha,N}(x) - (\varepsilon/2)(\alpha-2)x^{\alpha-2}\prod_{k=1}^{N}[\ln_k(x)]^{-1} + \big(\varepsilon^2/4\big)x^{\alpha-2}\prod_{k=1}^{N}[\ln_k(x)]^{-2} \\
+ \varepsilon x^{\alpha-2}\prod_{k=1}^{N}[\ln_k(x)]^{-1}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-1}, \qquad (3.34)
\end{aligned}
$$

$$
y_{N,\varepsilon}(x) = x^{-1/2}\prod_{k=1}^{N-1}[\ln_k(x)]^{-1/2}[\ln_N(x)]^{-1/2-\varepsilon/2}, \qquad (3.35)
$$

$$
\widetilde{y}_{N,\varepsilon}(x) = y_{N,\varepsilon}(x)\int_{x_0}^{x} dt\, t^{-\alpha} y_{N,\varepsilon}(t)^{-2}, \qquad (3.36)
$$

$$
\tau_{\alpha,N,\varepsilon} = -(d/dx)x^{\alpha}(d/dx) + q_{\alpha,N,\varepsilon}(x). \qquad (3.37)
$$

At this point we claim (cf. Lemma A.2 for details) that

$$
(\tau_{\alpha,N,\varepsilon}\, y_{N,\varepsilon})(x) = 0, \quad (\tau_{\alpha,N,\varepsilon}\, \widetilde{y}_{N,\varepsilon})(x) = 0, \quad \alpha \in (-\infty,2),\ N \in \mathbb{N}. \qquad (3.38)
$$

Since, for $0 < \delta_N$ sufficiently small,

$$
y_{N,\varepsilon} \in L^2((0,\delta_N); dx), \quad \alpha \in (-\infty,2),\ N \in \mathbb{N}, \qquad (3.39)
$$

and (utilizing $\alpha \in (-\infty,2)$)

$$
\int_0^{\delta_N} dt\, t^{1-\alpha}\left(\prod_{k=1}^{N-1}\ln_k(t)\right)[\ln_N(t)]^{1+\varepsilon} < \infty, \qquad (3.40)
$$

also

$$
\widetilde{y}_{N,\varepsilon} \in L^2((0,\delta_N); dx), \quad \alpha \in (-\infty,2),\ N \in \mathbb{N}. \qquad (3.41)
$$

Thus, $\tau_{\alpha,N,\varepsilon}$ is in the limit-circle case and nonoscillatory (cf. (3.35)) at $x = 0$. Moreover, since for $0 < \varepsilon$ and $0 < x$ sufficiently small,

$$
\begin{aligned}
q_{\alpha,N,\varepsilon}(x) \geq [(3/4) - (\alpha/2)]x^{\alpha-2} - (1/2)(2-\alpha)x^{\alpha-2}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-1} \\
- (\varepsilon/2)(2-\alpha)x^{\alpha-2}\prod_{k=1}^{N}[\ln_k(x)]^{-1}, \qquad (3.42)
\end{aligned}
$$

condition (3.26) implies that $q(x) \leq \widehat{Q}_{\alpha,N,\varepsilon}(x) \leq q_{\alpha,N,\varepsilon}(x)$ for $0 < \varepsilon$ and $0 < x$ sufficiently small. Thus, Theorem 2.9 implies that also τ_α, $\alpha \in (-\infty,2)$, is in the limit-circle case at $x = 0$. □

Remark 3.4. Thus far we have focused on endpoint classifications at $x = 0$. It is of course possible to address the limit-point case at $x = \infty$; however, this is typically based on a markedly different technique and we provide an example next.

[4] See footnote 3.

For this purpose we abbreviate iterated logarithms for $0 < x$ sufficiently large as follows:

$$\mathrm{Ln}_1(x) = \ln(x), \qquad \mathrm{Ln}_{j+1}(x) = \ln(\mathrm{Ln}_j(x)), \quad j \in \mathbb{N}. \tag{3.43}$$

Next, let $a \in \mathbb{R}$, $\alpha \in (-\infty, 2]$, and consider τ_α in (3.2) on the interval $[a, \infty)$. Suppose there exists $C \in (0, \infty)$ such that, for $R \in (a, \infty)$ sufficiently large (rendering $\mathrm{Ln}_\ell(\cdot)$, $1 \leq \ell \leq N$, strictly positive) and some $N \in \mathbb{N}$,

$$q(x) \geq -Cx^{2-\alpha} \prod_{k=1}^{N} [\mathrm{Ln}_k(x)]^2 \quad \text{for a.e. } x \in [R, \infty). \tag{3.44}$$

Then τ_α, $\alpha \in (-\infty, 2]$, is in the limit-point case at $x = \infty$.

For the proof it suffices to choose

$$p(x) = x^\alpha, \quad M(x) = x^{2-\alpha} \prod_{k=1}^{N} [\mathrm{Ln}_k(x)]^2, \quad x \in [R, \infty), \tag{3.45}$$

and refer to [6, Theorem 16, p. 1406]. (For a three-coefficient analogue of the latter, see [31, Theorem 7.4.3, pp. 148–149].)

For additional results regarding the absence of L^2-solutions at $x = \infty$, implying the limit-point case at $x = \infty$, we also refer the reader to [24], [25], and [26].

4. An elementary multi-dimensional application

In this section we briefly sketch an elementary multi-dimensional application of Theorem 3.1 in connection with the partial differential expression $-\operatorname{div}(p(|\cdot|)\nabla) + q(|\cdot|)$.

In n-dimensional spherical coordinates, the differential expression $-\operatorname{div}(p(|\cdot|)\nabla)$ on the n-dimensional ball $B_n(0; R) \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, $R \in (0, \infty)$, assuming

$$1/p \in L^1((\varepsilon, R); dr), \quad 0 < p \in AC([\varepsilon, R]) \text{ for all } \varepsilon > 0, \tag{4.1}$$

takes the form

$$-\operatorname{div} p(|x|)\nabla = -r^{1-n} \frac{\partial}{\partial r}\left(r^{n-1} p(r) \frac{\partial}{\partial r}\right) - \frac{p(r)}{r^2} \Delta_{\mathbb{S}^{n-1}}, \quad x \in B_n(0; R)\backslash\{0\}, \tag{4.2}$$

where $-\Delta_{\mathbb{S}^{n-1}}$ denotes the Laplace–Beltrami operator associated with the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. When acting in $L^2(B_n(0; R))$, which in spherical coordinates can be written as $L^2(B_n(0; R); d^n x) \simeq L^2((0, R); r^{n-1} dr) \otimes L^2(\mathbb{S}^{n-1})$, (4.2) becomes

$$-\operatorname{div} p(|x|)\nabla = \left[-\frac{d}{dr} p(r) \frac{d}{dr} - \frac{(n-1)p(r)}{r} \frac{d}{dr}\right] \otimes I_{L^2(\mathbb{S}^{n-1})} - \frac{p(r)}{r^2} \otimes \Delta_{\mathbb{S}^{n-1}} \tag{4.3}$$

(with $I_{\mathcal{X}}$ denoting the identity operator on $\mathcal{X}$). The Laplace–Beltrami operator $-\Delta_{\mathbb{S}^{n-1}}$ in $L^2(\mathbb{S}^{n-1})$, with domain $\operatorname{dom}(-\Delta_{\mathbb{S}^{n-1}}) = H^2(\mathbb{S}^{n-1})$ (cf., e.g., [2]), is known to be essentially self-adjoint and nonnegative on $C_0^\infty(\mathbb{S}^{n-1})$ (cf. [5, Theorem 5.2.3]). Recalling the treatment in [21, pp. 160–161], one decomposes the space $L^2(\mathbb{S}^{n-1})$ into an infinite orthogonal sum, yielding

$$\begin{aligned} L^2(B_n(0;R);d^nx) &\simeq L^2((0,R);r^{n-1}dr)\otimes L^2(\mathbb{S}^{n-1}) \\ &= \bigoplus_{\ell=0}^{\infty} L^2((0,R);r^{n-1}dr)\otimes \mathcal{Y}_\ell^n, \end{aligned} \tag{4.4}$$

where $\mathcal{Y}_\ell^n$ is the eigenspace of $-\Delta_{\mathbb{S}^{n-1}}$ corresponding to the eigenvalue $\ell(\ell+n-2)$, $\ell\in\mathbb{N}_0$, as

$$\sigma(-\Delta_{\mathbb{S}^{n-1}}) = \{\ell(\ell+n-2)\}_{\ell\in\mathbb{N}_0}. \tag{4.5}$$

In particular, this results in

$$-\operatorname{div} p(|x|)\nabla = \bigoplus_{\ell=0}^{\infty}\bigg[-\frac{d}{dr}p(r)\frac{d}{dr} - \frac{(n-1)p(r)}{r}\frac{d}{dr} + \frac{\ell(\ell+n-2)p(r)}{r^2}\bigg]\otimes I_{\mathcal{Y}_\ell^n}, \tag{4.6}$$

in the space (4.4).

To simplify matters, replacing the measure $r^{n-1}dr$ by dr and simultaneously removing the term $(n-1)p(r)r^{-1}(d/dr)$, one introduces the unitary operator

$$U_n = \begin{cases} L^2((0,R);r^{n-1}dr)\to L^2((0,R);dr), \\ f(r)\mapsto r^{(n-1)/2}f(r), \end{cases} \tag{4.7}$$

under which (4.6) becomes

$$\begin{aligned} -\operatorname{div} p(|x|)\nabla = \bigoplus_{\ell=0}^{\infty} U_n^{-1}\bigg[&-\frac{d}{dr}p(r)\frac{d}{dr} + (n-1)\frac{p'(r)}{2r} \\ &+ \{[(n-1)(n-3)/4] + \ell(\ell+n-2)\}\frac{p(r)}{r^2}\bigg]U_n\otimes I_{\mathcal{Y}_\ell^n}, \end{aligned} \tag{4.8}$$

still acting in the space (4.4). Thus, specializing to the case

$$p(r) = r^\alpha, \quad \alpha\in\mathbb{R},\ r\in(0,R], \tag{4.9}$$

the self-adjoint Friedrichs L^2-realization, $H_{\alpha,F}^{(0)}$, of $-\operatorname{div}|\cdot|^\alpha\nabla$ in the space (4.4) then is of the form

$$H_{\alpha,F}^{(0)} = \bigoplus_{\ell=0}^{\infty} U_n^{-1} h_{n,\ell,\alpha,F}^{(0)} U_n \otimes I_{\mathcal{Y}_\ell^n}, \tag{4.10}$$

where $h_{n,\ell,\alpha,F}^{(0)}$, $\ell\in\mathbb{N}_0$, represents the Friedrichs extension of the preminimal operator, $\dot{h}_{n,\ell,\alpha}^{(0)}$ in $L^2((0,R);dr)$, associated with the differential expression

$$\begin{aligned} \tau_{n,\ell,\alpha}^{(0)} = -\frac{d}{dr}r^\alpha\frac{d}{dr} + \frac{[(n-1)(n-3+2\alpha)/4] + \ell(\ell+n-2)}{r^{2-\alpha}}, \\ \alpha\in\mathbb{R},\ n\in\mathbb{N},\ n\geq 2,\ \ell\in\mathbb{N}_0,\ r\in(0,R], \end{aligned} \tag{4.11}$$

that is,

$$\overset{\bullet}{h}{}^{(0)}_{n,\ell,\alpha} = \tau^{(0)}_{n,\ell,\alpha}\big|_{C_0^\infty((0,R))}, \qquad \alpha \in \mathbb{R},\ n \in \mathbb{N},\ n \geq 2,\ \ell \in \mathbb{N}_0, \tag{4.12}$$

in $L^2((0,R);dr)$.

To explicitly describe $h^{(0)}_{n,\ell,\alpha,F}$ we next recall some results from [8] and [9]. For this purpose we introduce the differential expression

$$\tau_{\beta,\gamma} = \left[-\frac{d}{dx}x^\beta \frac{d}{dx} + \frac{(2-\beta)^2\gamma^2 - (1-\beta)^2}{4} x^{\beta-2}\right], \qquad \beta \in \mathbb{R},\ \gamma \in [0,\infty),\ x \in (0,R], \tag{4.13}$$

and recall that solutions to $\tau_{\beta,\gamma} y(z,\cdot) = zy(z,\cdot)$ are given by (cf. [16, No. 2.162, p. 440])

$$y_{1,\beta,\gamma}(z,x) = x^{(1-\beta)/2} J_\gamma\big(2z^{1/2}x^{(2-\beta)/2}/(2-\beta)\big), \quad \gamma \in [0,\infty), \tag{4.14}$$

$$y_{2,\beta,\gamma}(z,x) = \begin{cases} x^{(1-\beta)/2} J_{-\gamma}\big(2z^{1/2}x^{(2-\beta)/2}/(2-\beta)\big), & \gamma \notin \mathbb{N}_0, \\ x^{(1-\beta)/2} Y_\gamma\big(2z^{1/2}x^{(2-\beta)/2}/(2-\beta)\big), & \gamma \in \mathbb{N}_0, \end{cases} \quad \gamma \in [0,\infty), \quad x \in (0,R], \tag{4.15}$$

where $J_\nu(\cdot)$, $Y_\nu(\cdot)$ are the standard Bessel functions of order $\nu \in \mathbb{R}$ (cf. [1, Ch. 9]). Solutions for $z = 0$ are particularly simple and we note that (non-normalized) principal and nonprincipal solutions $u_{0,\beta,\gamma}(0,\cdot)$ and $\widehat{u}_{0,\beta,\gamma}(0,\cdot)$ of $\tau_{\beta,\gamma} u = 0$ at $x = 0$ are of the form

$$\begin{aligned} &u_{0,\beta,\gamma}(0,x) = x^{[1-\beta+(2-\beta)\gamma]/2}, \quad \gamma \in [0,\infty), \\ &\widehat{u}_{0,\beta,\gamma}(0,x) = \begin{cases} x^{[1-\beta-(2-\beta)\gamma]/2}, & \gamma \in (0,\infty), \\ x^{(1-\beta)/2}\ln(1/x), & \gamma = 0, \end{cases} \\ &\qquad\qquad \beta \in \mathbb{R},\ x \in (0,1), \\ &\widehat{u}_{0,2,\gamma}(0,x) = x^{-1/2}\ln(1/x), \quad \gamma \in [0,\infty),\ x \in (0,1). \end{aligned} \tag{4.16}$$

In particular, $\tau_{\beta,\gamma}$, $\beta \in \mathbb{R}$, $\gamma \in [0,\infty)$, is nonoscillatory at $x = 0$ and $x = R$, regular at $x = R$, and the following limit-point/limit-circle classification holds:

$$\begin{cases} \tau_{\beta,\gamma} \text{ is in the limit-point case at } x = 0 \text{ if } \beta \in [2,\infty),\ \gamma \in [0,\infty) \\ \qquad \text{and if } \beta \in (-\infty,2),\ \gamma \in [1,\infty), \\ \tau_{\beta,\gamma} \text{ is in the limit-circle case at } x = 0 \text{ if } \beta \in (-\infty,2),\ \gamma \in [0,1), \\ \tau_{\beta,\gamma} \text{ is in the limit-circle case at } x = R \text{ if } \beta \in \mathbb{R},\ \gamma \in [0,\infty). \end{cases} \tag{4.17}$$

The preminimal, $\dot{T}_{\beta,\gamma}$, and maximal, $T_{\beta,\gamma,max}$, $L^2((0,R];dx)$-realizations associated with $\tau_{\beta,\gamma}$, $\beta \in \mathbb{R}$, $\gamma \in [0,\infty)$, are then given by

$$\dot{T}_{\beta,\gamma} = \tau_{\beta,\gamma}\big|_{C_0^\infty((0,R))}, \tag{4.18}$$

$$\begin{aligned}
&(T_{\beta,\gamma,max}f)(x) = (\tau_{\beta,\gamma}f)(x) \quad \text{for a.e. } x \in (0,R], \\
&f \in \operatorname{dom}(T_{\beta,\gamma,max}) = \big\{g \in L^2((0,R);dx) \,\big|\, g, g' \in AC_{\text{loc}}((\varepsilon,R]) \text{ for all } 0<\varepsilon<R; \\
&\qquad\qquad\qquad\qquad\qquad\qquad \tau_{\beta,\gamma}g \in L^2((0,R);dx)\big\}.
\end{aligned} \tag{4.19}$$

According to [8], the generalized boundary values for $g \in \operatorname{dom}(T_{\beta,\gamma,max})$ at $x=0$ in the limit-circle case at $x=0$ (i.e., if $\beta \in (-\infty,2)$, $\gamma \in [0,1)$) are of the form

$$\widetilde{g}(0) = \begin{cases} \lim_{x\downarrow 0} g(x)\big/\big[x^{[1-\beta-(2-\beta)\gamma]/2}\big], & \gamma \in (0,1), \\ \lim_{x\downarrow 0} g(x)\big/\big[x^{(1-\beta)/2}\ln(1/x)\big], & \gamma = 0, \end{cases} \tag{4.20}$$

$$\widetilde{g}\,'(0) = \begin{cases} \lim_{x\downarrow 0} \big[g(x) - \widetilde{g}(0)x^{[1-\beta-(2-\beta)\gamma]/2}\big]\big/\big[x^{[1-\beta+(2-\beta)\gamma]/2}\big], & \gamma \in (0,1), \\ \lim_{x\downarrow 0} \big[g(x) - \widetilde{g}(0)x^{(1-\beta)/2}\ln(1/x)\big]\big/\big[x^{(1-\beta)/2}\big], & \gamma = 0. \end{cases} \tag{4.21}$$

Since $\tau_{\beta,\gamma}$ is regular at $x=R$, the standard boundary values for $g \in \operatorname{dom}(T_{\beta,\gamma,max})$ at $x=R$ are of the standard form $g(R)$, $g'(R)$.

The closure of $\dot{T}_{\beta,\gamma}$ in $L^2((0,R);dx)$, that is, the minimal operator, $T_{\beta,\gamma,min}$, associated with $\tau_{\beta,\gamma}$, is then given by

$$\begin{aligned}
&(T_{\beta,\gamma,min}f)(x) = (\tau_{\beta,\gamma}f)(x) \quad \text{for a.e. } x \in (0,R],\ \beta \in \mathbb{R},\ \gamma \in [0,\infty), \\
&f \in \operatorname{dom}(T_{\beta,\gamma,min}) = \big\{g \in \operatorname{dom}(T_{\beta,\gamma,max}) \,\big|\, \widetilde{g}(0) = \widetilde{g}\,'(0) = 0,\ g(R) = g'(R) = 0\big\}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \beta \in (-\infty,2),\ \gamma \in [0,1),
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&f \in \operatorname{dom}(T_{\beta,\gamma,min}) = \big\{g \in \operatorname{dom}(T_{\beta,\gamma,max}) \,\big|\, g(R) = g'(R) = 0\big\}, \\
&\qquad \beta \in (-\infty,2),\ \gamma \in [1,\infty),\ \text{or},\ \beta \in [2,\infty),\ \gamma \in [0,\infty),
\end{aligned} \tag{4.23}$$

and the Friedrichs extension, $T_{\beta,\gamma,F}$, of $T_{\beta,\gamma,min}$ (and $\dot{T}_{\beta,\gamma}$) is characterized by (cf. [13], [19], [22])

$$\begin{aligned}
&(T_{\beta,\gamma,F}f)(x) = (\tau_{\beta,\gamma}f)(x) \quad \text{for a.e. } x \in (0,R],\ \beta \in \mathbb{R},\ \gamma \in [0,\infty), \\
&f \in \operatorname{dom}(T_{\beta,\gamma,F}) = \big\{g \in \operatorname{dom}(T_{\beta,\gamma,max}) \,\big|\, \widetilde{g}(0) = 0,\ g(R) = 0\big\}, \\
&\qquad\qquad\qquad\qquad \beta \in (-\infty,2),\ \gamma \in [0,1),
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
&f \in \operatorname{dom}(T_{\beta,\gamma,F}) = \big\{g \in \operatorname{dom}(T_{\beta,\gamma,max}) \,\big|\, g(R) = 0\big\}, \\
&\qquad \beta \in (-\infty,2),\ \gamma \in [1,\infty),\ \text{or},\ \beta \in [2,\infty),\ \gamma \in [0,\infty).
\end{aligned} \tag{4.25}$$

Returning to $\tau^{(0)}_{n,\ell,\alpha}$, and hence comparing

$$(n-1)(n-3+2\alpha)+4\ell(\ell+n-2) \quad \text{with} \quad (2-\alpha)^2\gamma^2-(1-\alpha)^2 \quad \text{for } \alpha \in \mathbb{R}\backslash\{0\}, \tag{4.26}$$

and treating the case $\alpha = 2$ separately, an application of (4.13)–(4.25) then yields the following facts for its limit-point/limit-circle classification, for the maximal operator $h^{(0)}_{n,\ell,\alpha,max}$ associated with $\tau^{(0)}_{n,\ell,\alpha}$, and the Friedrichs extension $h^{(0)}_{n,\ell,\alpha,F}$ of $\overset{\bullet}{h}{}^{(0)}_{n,\ell,\alpha}$ in $L^2((0,R);dx)$. First, upon identifying

$$\beta = \alpha \in \mathbb{R}\backslash\{2\}, \quad \gamma = \gamma_\alpha = \big[(2-\alpha-n)^2 + 4\ell(\ell+n-2)\big]^{1/2}/|2-\alpha| \in [0,\infty), \tag{4.27}$$

more precisely,

$$\begin{aligned} &\alpha > 2, \quad \gamma_\alpha \in [1,\infty), \\ &\alpha < 2, \quad \gamma_\alpha \in [0,\infty), \end{aligned} \tag{4.28}$$

$\tau^{(0)}_{n,\ell,\alpha}$, $\alpha \in \mathbb{R}$, is nonoscillatory at $r=0$ and $r=R$, and regular at $r=R$. In addition,

$$\begin{cases} \tau^{(0)}_{n,\ell,\alpha} \text{ is in the limit-point case at } r=0 \text{ if } \alpha \in (2,\infty),\ \gamma_\alpha \in [0,\infty), \\ \qquad \text{if } \alpha \in (-\infty,2),\ \gamma_\alpha \in [1,\infty), \text{ and if } \alpha = 2; \\ \tau^{(0)}_{n,\ell,\alpha} \text{ is in the limit-circle case at } r=0 \text{ if } \alpha \in (-\infty,2),\ \gamma_\alpha \in [0,1); \\ \tau^{(0)}_{n,\ell,\alpha} \text{ is in the limit-circle case at } r=R \text{ for all } \alpha \in \mathbb{R}; \end{cases} \tag{4.29}$$

and hence

$$\begin{cases} \tau^{(0)}_{n,\ell,\alpha} \text{ is in the limit-point case at } r=0 \text{ if and only if} \\ \qquad \alpha \in [2-(n/2)-(2/n)\ell(\ell+n-2),\infty), \\ \tau^{(0)}_{n,\ell,\alpha} \text{ is in the limit-circle case at } r=0 \text{ if and only if} \\ \qquad \alpha \in (-\infty, 2-(n/2)-(2/n)\ell(\ell+n-2)), \\ \tau^{(0)}_{n,\ell,\alpha} \text{ is in the limit-circle case at } r=R \text{ for all } \alpha \in \mathbb{R}. \end{cases} \tag{4.30}$$

Moreover, the underlying maximal operator is of the form

$$\begin{aligned} &\big(h^{(0)}_{n,\ell,\alpha,max} f\big)(r) = \big(\tau^{(0)}_{n,\ell,\alpha} f\big)(r) \quad \text{for a.e. } r \in (0,R],\ \alpha \in \mathbb{R}, \\ &f \in \operatorname{dom}(h^{(0)}_{n,\ell,\alpha,max}) = \big\{g \in L^2((0,R);dr) \,\big|\, g, g' \in AC([\varepsilon,R]) \text{ for all } \varepsilon \in (0,R); \\ &\qquad\qquad\qquad\qquad\qquad\qquad \big(\tau^{(0)}_{n,\ell,\alpha} f\big)(r) \in L^2((0,R);dr)\big\}, \end{aligned} \tag{4.31}$$

and the corresponding Friedrichs extension of $\overset{\bullet}{h}{}^{(0)}_{n,\ell,\alpha}$ is given by

$$\begin{aligned} &\big(h^{(0)}_{n,\ell,\alpha,F} f\big)(r) = \big(\tau^{(0)}_{n,\ell,\alpha} f\big)(r) \text{ for a.e. } r \in (0,R],\ \alpha \in \mathbb{R}, \\ &f \in \operatorname{dom}(h^{(0)}_{n,\ell,\alpha,F}) = \big\{g \in \operatorname{dom}(h^{(0)}_{n,\ell,\alpha,max}) \,\big|\, \widetilde{g}(0) = 0,\ g(R) = 0\big\}, \\ &\qquad\qquad\qquad \alpha \in (-\infty, 2-(n/2)-(2/n)\ell(\ell+n-2)), \end{aligned} \tag{4.32}$$

$$\begin{aligned} &f \in \operatorname{dom}(h^{(0)}_{n,\ell,\alpha,F}) = \big\{g \in \operatorname{dom}(h^{(0)}_{n,\ell,\alpha,max}) \,\big|\, g(R) = 0\big\}, \\ &\qquad\qquad\qquad \alpha \in [2-(n/2)-(2/n)\ell(\ell+n-2),\infty). \end{aligned} \tag{4.33}$$

Here the boundary value $\widetilde{g}(0)$ associated with $g \in \operatorname{dom}\big(h^{(0)}_{n,\ell,\alpha,max}\big)$, $n \in \mathbb{N}$, $n \geq 2$, $\ell \in \mathbb{N}_0$, $\alpha \in (-\infty, 2)$, is now given by

$$\begin{aligned}\widetilde{g}(0) &= \begin{cases} \lim_{x\downarrow 0} g(x)\big/\big[x^{[1-\alpha-(2-\alpha)\gamma_\alpha]/2}\big], & \gamma_\alpha \in (0,1),\\ \lim_{x\downarrow 0} g(x)\big/\big[x^{(1-\alpha)/2}\ln(1/x)\big], & \gamma_\alpha = 0, \end{cases}\\ &\qquad\qquad\qquad\qquad \alpha \in (-\infty, 2),\\ &= \begin{cases} \lim_{x\downarrow 0} g(x)\big/\big[x^{\{1-\alpha-[(2-\alpha-n)^2+4\ell(\ell+n-2)]^{1/2}\}/2}\big],\\ \qquad \alpha \in (-\infty, 2-(n/2)-(2/n)\ell(\ell+n-2)),\ \ell \in \mathbb{N}_0,\\ \lim_{x\downarrow 0} g(x)\big/\big[x^{(1-\alpha)/2}\ln(1/x)\big], \quad \alpha = 2-n,\ \ell = 0. \end{cases}\end{aligned} \tag{4.34}$$

Without going into details, we note that utilizing the transformation (4.7), and invoking results of Kalf [12], the operator $H^{(0)}_{\alpha,F}$ is of the following form:

$$\begin{aligned}&\big(H^{(0)}_{\alpha,F}\psi\big)(x) = -(\operatorname{div}|x|^\alpha \nabla\psi)(x), \quad x \in B_n(0;R)\backslash\{0\},\\ &\psi \in \operatorname{dom}\big(H^{(0)}_{\alpha}\big) = \bigg\{\phi \in \operatorname{dom}\big(H^{(0)}_{\alpha,max}\big)\,\bigg|\, |\cdot|^\alpha(\nabla\phi) \in L^2(B_n(0;R);d^nx);\\ &\qquad \lim_{r\uparrow R}\int_{\mathbb{S}^{n-1}} d^{n-1}\omega\,|\phi(r\omega)|^2 = 0,\\ &\qquad \text{and if and only if } n < 2-\alpha,\ \lim_{r\downarrow 0}\int_{\mathbb{S}^{n-1}} d^{n-1}\omega\,|\phi(r\omega)|^2 = 0\bigg\}\end{aligned} \tag{4.35}$$

(with $d^{n-1}\omega$ the surface measure on $\mathbb{S}^{n-1}$), where

$$\begin{aligned}&\big(H^{(0)}_{\alpha,max}\psi\big)(x) = -(\operatorname{div}|x|^\alpha\nabla\psi)(x), \quad x \in B_n(0;R)\backslash\{0\},\\ &\psi \in \operatorname{dom}\big(H^{(0)}_{\alpha,max}\big) = \big\{\phi \in L^2(B_n(0;R);d^nx)\,\big|\,\phi \in H^2_{\rm loc}(B_n(0;R)\backslash\{0\});\\ &\qquad\qquad \operatorname{div}|\cdot|^\alpha\nabla\phi \in L^2(B_n(0;R);d^nx)\big\}.\end{aligned} \tag{4.36}$$

We remark that the boundary condition at $x = R$ (and at $x = 0$ if and only if $n < 2 - \alpha$) has to be imposed on a distinguished representative of ϕ for which the restriction to the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ exists as a square-integrable function (see the discussion in [12, Remark 3]).

We conclude these considerations by adding an additional potential term q in accordance with inequalities (3.4) and (3.5). For this purpose we assume that, for all $\eta \in (0, R)$,

$$q \in L^1((\eta, R); dr) \text{ is real-valued a.e. on } (0, R), \tag{4.37}$$

and introduce, for $n \in \mathbb{N}$, $n \geq 2$, $\ell \in \mathbb{N}_0$,

$$\tau_{n,\ell,\alpha} = \tau^{(0)}_{n,\ell,\alpha} + q(r) \quad \text{for a.e. } r \in (0,R),\ \alpha \in \mathbb{R}, \tag{4.38}$$

and the following $L^2((0,R);dr)$-realization of $\tau_{n,\ell,\alpha}$:

$$\begin{aligned}&\big(h_{n,\ell,\alpha}f\big)(r) = \big(\tau_{n,\ell,\alpha}f\big)(r) \quad \text{for a.e. } r \in (0,R],\ \alpha \in \mathbb{R},\\ &f \in \operatorname{dom}(h_{n,\ell,\alpha}) = \big\{g \in L^2((0,R);dr)\,\big|\, g, g' \in AC([\varepsilon,R]) \text{ for all } \varepsilon \in (0,R);\\ &\qquad\qquad g(R) = 0;\ \big(\tau^{(0)}_{n,\ell,\alpha}f\big)(r) \in L^2((0,R);dr)\big\}.\end{aligned} \tag{4.39}$$

Theorem 4.1. *Assume* $n \in \mathbb{N}$, $n \geq 2$, *and* (4.37).

(i) *If* $\alpha \in \mathbb{R}$, $\ell \in \mathbb{N}_0$, *and, for a.e.* $0 < r$ *sufficiently small,*

$$q(r) \geq -\{[n(n-4+2\alpha)/4] + \ell(\ell+n-2)\} r^{\alpha-2}, \tag{4.40}$$

then $\tau_{n,\ell,\alpha}$ *is nonoscillatory and in the limit-point case at* $r = 0$.

(ii) *If*[5] $\alpha \in (-\infty, 2)$, $\ell \in \mathbb{N}_0$, *and there exist* $N \in \mathbb{N}$ *and* $\varepsilon > 0$ *such that, for a.e.* $0 < r$ *sufficiently small (depending on* N *and* ε*),*

$$\begin{aligned} q(r) \geq & -\{[n(n-4+2\alpha)/4] + \ell(\ell+n-2)\} r^{\alpha-2} \\ & - (1/2)(2-\alpha) r^{\alpha-2} \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(r)]^{-1} + [(3/4)+\varepsilon] r^{\alpha-2} [\ln_1(x)]^{-2}, \end{aligned} \tag{4.41}$$

then $\tau_{n,\ell,\alpha}$ *is nonoscillatory and in the limit-point case at* $r = 0$.

(iii) *Assuming inequality* (4.40) *(for* $\alpha \in \mathbb{R}$*) or* (4.41) *(for* $\alpha \in (-\infty, 2)$*) holds for* $\ell = 0$, *then the operator*

$$H_\alpha = \bigoplus_{\ell \in \mathbb{N}_0} U_n^{-1} h_{n,\ell,\alpha} U_n \tag{4.42}$$

is self-adjoint in $L^2(B_n(0;R); d^n x)$.

Proof. Items (i) and (ii) are an immediate consequence of Theorem 3.1 since, for instance, inequality (4.40) in the context $N = 0$ is equivalent to

$$\{[(n-1)(n-3+2\alpha)/4] + \ell(\ell+n-2)\} r^{\alpha-2} + q(r) \geq [(3/4) - (\alpha/2)] r^{\alpha-2} \tag{4.43}$$

for $0 < r$ sufficiently small (cf. (3.4)), and analogously in the context of (4.41) (cf. (3.5)) for $N \in \mathbb{N}$. Item (iii) holds since $h_{n,0,\alpha}$ being self-adjoint in $L^2((0,R); dr)$ implies that $h_{n,\ell,\alpha}$ is self-adjoint in $L^2((0,R); dr)$ for all $\ell \in \mathbb{N}_0$. □

In the case $N = 0$, the self-adjointness of H_α is familiar from multi-dimensional results by Kalf and Walter [15] (with strict inequality in the analogue of (4.40) for $\ell = 0$); in this context, see also [14], [23] for $\alpha = 0$.

Appendix A. More details in connection with Theorems 3.1 and 3.3

In this appendix we elaborate on the proofs of Theorems 3.1 and 3.3 by sketching the proofs of the assertions in (3.14) and (3.38).

We begin with the limit-point case discussed in Theorem 3.1.

Lemma A.1. *Let the assumptions of Theorem* 3.1 *be satisfied, and let* $q_{\alpha,N}(x)$, $\tau_{\alpha,N}$, *and* $y_N(x)$ *be as in* (3.11)–(3.13). *Then, for all* $N \in \mathbb{N}$,

$$(\tau_{\alpha,N}\, y_N)(x) = 0. \tag{A.1}$$

[5] Again, only $\alpha \in (-\infty, 2)$ can improve on item (i).

Proof. One observes[6]

$$\big(\ln_N(x)\big)' = -x^{-1}\prod_{k=1}^{N-1}[\ln_k(x)]^{-1}, \tag{A.2}$$

$$\Big([\ln_N(x)]^{-1/2}\Big)' = \frac{1}{2}x^{-1}\left(\prod_{k=1}^{N-1}[\ln_k(x)]^{-1}\right)[\ln_N(x)]^{-3/2}, \tag{A.3}$$

$$\left(\prod_{\ell=1}^{N}[\ln_\ell(x)]^{-1/2}\right)' = \frac{1}{2}x^{-1}\prod_{k=1}^{N}[\ln_k(x)]^{-1/2}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-1}, \tag{A.4}$$

$$\left(\prod_{k=1}^{N}[\ln_k(x)]^{-1}\right)' = x^{-1}\prod_{k=1}^{N}[\ln_k(x)]^{-1}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-1}, \tag{A.5}$$

and hence verifies

$$\begin{aligned}
&\frac{1}{y_N(x)}\big(x^\alpha y_N'(x)\big)' \\
&\quad= \frac{1}{y_N(x)}\frac{d}{dx}\Bigg[-\frac{1}{2}x^{\alpha-3/2}\prod_{k=1}^{N}[\ln_k(x)]^{-1/2} \\
&\qquad+ \frac{1}{2}x^{\alpha-3/2}\prod_{k=1}^{N}[\ln_k(x)]^{-1/2}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-1}\Bigg] \\
&\quad= \frac{1}{y_N(x)}\frac{d}{dx}\left[\frac{1}{2}x^{\alpha-3/2}\prod_{k=1}^{N}[\ln_k(x)]^{-1/2}\left(-1+\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-1}\right)\right] \\
&\quad= \frac{1}{y_N(x)}\Bigg[\frac{1}{2}\left(\alpha-\frac{3}{2}\right)x^{\alpha-5/2}\prod_{k=1}^{N}[\ln_k(x)]^{-1/2}\left(-1+\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-1}\right) \\
&\qquad+ \frac{1}{4}x^{\alpha-5/2}\prod_{k=1}^{N}[\ln_k(x)]^{-1/2}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-1}\left(-1+\sum_{m=1}^{N}\prod_{p=1}^{m}[\ln_p(x)]^{-1}\right) \\
&\qquad+ \frac{1}{2}x^{\alpha-5/2}\prod_{k=1}^{N}[\ln_k(x)]^{-1/2}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-1}\sum_{m=1}^{j}\prod_{p=1}^{m}[\ln_p(x)]^{-1}\Bigg] \\
&\quad= \left(\frac{3}{4}-\frac{\alpha}{2}\right)x^{\alpha-2} + \left(\frac{\alpha}{2}-\frac{3}{4}\right)x^{\alpha-2}\sum_{j=1}^{N}\prod_{\ell=1}^{j}[\ln_\ell(x)]^{-1}
\end{aligned}$$

[6] See footnote 3.

$$- \frac{1}{4} x^{\alpha-2} \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1} + \frac{1}{4} x^{\alpha-2} \Bigg(\sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1} \Bigg)^2$$
$$+ \frac{1}{2} x^{\alpha-2} \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1} \sum_{m=1}^{j} \prod_{p=1}^{m} [\ln_p(x)]^{-1}. \tag{A.6}$$

Using the equality

$$\sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1} \sum_{m=1}^{j} \prod_{p=1}^{m} [\ln_p(x)]^{-1} = \sum_{j=1}^{N-1} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-2} \sum_{m=j+1}^{N} \prod_{p=j+1}^{m} [\ln_p(x)]^{-1}$$
$$+ \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-2}, \tag{A.7}$$

one rewrites the last line in (A.6) in the form

$$\frac{1}{4} x^{\alpha-2} \Bigg(\sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1} \Bigg)^2 + \frac{1}{2} x^{\alpha-2} \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1} \sum_{m=1}^{j} \prod_{p=1}^{m} [\ln_p(x)]^{-1}$$
$$= \frac{1}{4} x^{\alpha-2} \sum_{j=1}^{N-1} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1} \sum_{m=j+1}^{N} \prod_{p=1}^{m} [\ln_p(x)]^{-1}$$
$$+ \frac{3}{4} x^{\alpha-2} \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1} \sum_{m=1}^{j} \prod_{p=1}^{m} [\ln_p(x)]^{-1} \tag{A.8}$$
$$= x^{\alpha-2} \sum_{j=1}^{N-1} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-2} \sum_{m=j+1}^{N} \prod_{p=j+1}^{m} [\ln_p(x)]^{-1} + \frac{3}{4} x^{\alpha-2} \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-2}.$$

Taking into account (3.11), (A.6), and (A.8), one derives the result

$$\frac{1}{y_N(x)} \big(x^\alpha y_N'(x) \big)' = [(3/4) - (\alpha/2)] x^{\alpha-2} - (1/2)(2-\alpha) x^{\alpha-2} \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-1}$$
$$+ (3/4) x^{\alpha-2} \sum_{j=1}^{N} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-2} = q_{\alpha,N}(x)$$
$$+ x^{\alpha-2} \sum_{j=1}^{N-1} \prod_{\ell=1}^{j} [\ln_\ell(x)]^{-2} \sum_{m=j+1}^{N} \prod_{p=j+1}^{m} [\ln_p(x)]^{-1}. \tag{A.9}$$

□

In connection with the limit-circle case discussed in Theorem 3.3 we note the following result:

Lemma A.2. *Let the conditions of Theorem* 3.3 *be satisfied, and let* $q_{\alpha,N,\varepsilon}(x)$*,* $\tau_{\alpha,N,\varepsilon}$*,* $y_{N,\varepsilon}(x)$*, and* $\widetilde{y}_{N,\varepsilon}(x)$ *be as in* (3.34)*–*(3.37)*. Then, for all* $N \in \mathbb{N}$*,*

$$(\tau_{\alpha,N,\varepsilon}\, y_{N,\varepsilon})(x) = 0, \qquad (\tau_{\alpha,N,\varepsilon}\, \widetilde{y}_{N,\varepsilon})(x) = 0. \tag{A.10}$$

Proof. Since $y_{N,\varepsilon}(x) = y_N(x)[\ln_N(x)]^{-\varepsilon/2}$, one derives

$$\begin{aligned}\big(x^\alpha y'_{N,\varepsilon}(x)\big)' &= \big(x^\alpha y'_N(x)\big)'[\ln_N(x)]^{-\varepsilon/2} + 2x^\alpha y'_N(x)\big([\ln_N(x)]^{-\varepsilon/2}\big)' \\ &\quad + x^\alpha y_N(x)\big([\ln_N(x)]^{-\varepsilon/2}\big)'' + \alpha x^{\alpha-1} y_N(x)\big([\ln_N(x)]^{-\varepsilon/2}\big)'.\end{aligned} \tag{A.11}$$

Employing (A.2), (A.4), and (A.5), we next calculate several derivatives, which appear on the right-hand side of (A.11):

$$\big([\ln_N(x)]^{-\varepsilon/2}\big)' = (\varepsilon/2)x^{-1}[\ln_N(x)]^{-\varepsilon/2}\prod_{k=1}^N [\ln_k(x)]^{-1}, \tag{A.12}$$

$$\begin{aligned}\big([\ln_N(x)]^{-\varepsilon/2}\big)'' &= -(\varepsilon/2)x^{-2}[\ln_N(x)]^{-\varepsilon/2}\prod_{k=1}^N [\ln_k(x)]^{-1} \\ &\quad + \big(\varepsilon^2/4\big)x^{-2}[\ln_N(x)]^{-\varepsilon/2}\prod_{k=1}^N [\ln_k(x)]^{-2} \\ &\quad + (\varepsilon/2)x^{-2}[\ln_N(x)]^{-\varepsilon/2}\prod_{k=1}^N [\ln_k(x)]^{-1}\sum_{j=1}^N\prod_{\ell=1}^j [\ln_\ell(x)]^{-1},\end{aligned} \tag{A.13}$$

$$\frac{y'_N(x)}{y_N(x)} = (2x)^{-1}\Bigg[-1 + \sum_{j=1}^N \prod_{\ell=1}^j [\ln_\ell(x)]^{-1}\Bigg]. \tag{A.14}$$

It follows from the last equality in (A.9), from (A.11), from (A.12)–(A.14), and from (3.34) that

$$\begin{aligned}&\frac{1}{y_{N,\varepsilon}(x)}\big(x^\alpha y'_{N,\varepsilon}(x)\big)' = q_{\alpha,N}(x) - (\varepsilon/2)x^{\alpha-2}\prod_{k=1}^N [\ln_k(x)]^{-1}\Bigg(1 - \sum_{j=1}^N\prod_{\ell=1}^j [\ln_\ell(x)]^{-1}\Bigg) \\ &\quad + (\varepsilon/2)x^{\alpha-2}\Bigg(-\prod_{k=1}^N [\ln_k(x)]^{-1} + (\varepsilon/2)\prod_{k=1}^N [\ln_k(x)]^{-2} \\ &\quad + \prod_{k=1}^N [\ln_k(x)]^{-1}\sum_{j=1}^N\prod_{\ell=1}^j [\ln_\ell(x)]^{-1}\Bigg) + (\varepsilon/2)\alpha x^{\alpha-2}\prod_{k=1}^N [\ln_k(x)]^{-1} \\ &= q_{\alpha,N}(x) - (\varepsilon/2)(2-\alpha)x^{\alpha-2}\prod_{k=1}^N [\ln_k(x)]^{-1} + \big(\varepsilon^2/4\big)x^{\alpha-2}\prod_{k=1}^N [\ln_k(x)]^{-2} \\ &\quad + \varepsilon x^{\alpha-2}\prod_{k=1}^N [\ln_k(x)]^{-1}\sum_{j=1}^N\prod_{\ell=1}^j [\ln_\ell(x)]^{-1} \\ &= q_{\alpha,N,\varepsilon}(x).\end{aligned} \tag{A.15}$$

Thus, the first equality in (A.10) is proved, and the second equality in (A.10) is clear from the reduction of order approach. □

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