

BOUNDEDNESS OF FRACTIONAL OPERATORS ASSOCIATED WITH SCHRÖDINGER OPERATORS ON WEIGHTED VARIABLE LEBESGUE SPACES VIA EXTRAPOLATION

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ABSTRACT. In this work we obtain boundedness results for fractional operators associated with Schrödinger operators $\mathcal{L} = -\Delta + V$ on weighted variable Lebesgue spaces. These operators include fractional integrals and their respective commutators. In particular, we obtain weighted inequalities of the type $L^{p(\cdot)}-L^{q(\cdot)}$ and estimates of the type $L^{p(\cdot)}$ -Lipschitz variable integral spaces. For this purpose, we developed extrapolation results that allow us to obtain boundedness results of the type described above in the variable setting by starting from analogous inequalities in the classical context. Such extrapolation results generalize what was done by Harboure, Macías, and Segovia [*Amer. J. Math.* **110** no. 3 (1988), 383–397], and by Bongioanni, Cabral, and Harboure [*Potential Anal.* **38** no. 4 (2013), 1207–1232], for the classic case, that is, $V \equiv 0$ and $p(\cdot)$ constant, respectively.

1. INTRODUCTION

Let us consider the second-order Schrödinger differential operator in \mathbb{R}^n with $n \geq 3$, defined by

$$\mathcal{L} = -\Delta + V,$$

where $V \geq 0$ and belongs to a reverse-Hölder class RH_ν , for some exponent $\nu \geq n/2$, i.e., there exists a constant C such that

$$\left(\frac{1}{|B|} \int_B V(x)^\nu dx \right)^{1/\nu} \leq \frac{C}{|B|} \int_B V(x) dx,$$

for every ball $B \subset \mathbb{R}^n$.

In recent years, a wide range of operators associated to \mathcal{L} have been catching the attention of several authors (see [1, 5, 3, 4, 18, 17, 26, 27]). One of those is the

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fractional integral, which is defined by means of negative fractional powers of \mathcal{L} , i.e.,

$$\mathcal{I}_\alpha = \mathcal{L}^{-\alpha/2},$$

for $0 < \alpha < n$.

In [17] the authors proved that \mathcal{I}_α is bounded from the Lebesgue space L^p to L^q , provided that $1/q = 1/p - n/\alpha$ and $1 < p < n/\alpha$, as the classical fractional integral. The corresponding weighted inequality was demonstrated in [3]. Since the negative powers of \mathcal{L} can be expressed in terms of the heat diffusion semigroup generated by \mathcal{L} , the kernel of \mathcal{I}_α has better behavior far away from the diagonal than the classical one. Thus, the family of weights $A_{p,q}^\rho$ (see Section 3) obtained in the boundedness $L^p(w^p) \rightarrow L^q(w^q)$ is wider than the class of weights of Muckenhoupt $A_{p,q}$ associated to the classical operator.

For the limiting case $p = n/\alpha$, in [1] they characterize the weights w such that \mathcal{I}_α is bounded from $L^{n/\alpha}(w^{n/\alpha})$ into $BMO_{\mathcal{L}}(w)$, an appropriate weighted version of BMO , the John–Nirenberg space adapted to this context.

The case of the commutator $[\mathcal{I}_\alpha, b]$ of \mathcal{I}_α was treated in [27] where it was shown that it is bounded from $L^p(w^p)$ into $L^q(w^q)$, for $1/q = 1/p - n/\alpha$, $1 < p < n/\alpha$ and $w \in A_{p,q}^\rho$.

In this context, maximal operators of fractional type can also be considered. Its definition is closely connected to the operator \mathcal{L} through the critical radius function

$$\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V \leq 1 \right\}.$$

In particular, the reverse-Hölder condition implies that $0 < \rho(x) < \infty$ for $x \in \mathbb{R}^n$. Furthermore, according to [26, Lemma 1.4], if $V \in RH_\nu$, with $\nu > n/2$, the associated function ρ satisfies that there exist constants $c_\rho, N_\rho \geq 1$ such that

$$c_\rho^{-1} \rho(x) \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N_\rho} \leq \rho(y) \leq c_\rho \rho(x) \left(1 + \frac{|x-y|}{\rho(x)} \right)^{\frac{N_\rho}{N_\rho+1}}, \quad (1.1)$$

for every $x, y \in \mathbb{R}^n$.

Shen uses this function ρ as an essential tool to obtain boundedness on L^p of many operators associated with \mathcal{L} , for instance the Riesz Schrödinger transform $\mathcal{R} = \nabla(-\Delta + V)^{-1/2}$. The point is that when \mathcal{R} is restricted to the ρ -neighborhood of the diagonal $\Delta_\rho = \{(x, y) : |x - y| < \rho(x)\}$, it behaves like the classical Riesz transform $R = \nabla(-\Delta)^{-1/2}$, in the sense that the difference of their kernels has a locally integrable singularity at the diagonal. Usually such restriction is called the ρ -local part of the kernel. As for the other part, the presence of the potential V guarantees a good decay scaled according to ρ .

This function is used to describe functional spaces and weights naturally associated with \mathcal{L} that are generalizations of the case $V \equiv 0$.

One of the main objectives of this work is to establish, by means of extrapolation techniques, boundedness results for fractional operators associated with Schrödinger operators on weighted variable Lebesgue spaces.

In reference to the theory of extrapolation, in [19], Rubio de Francia proved that the classes A_p satisfy an interesting property of extrapolation. More specifically, if for some $p_0 \geq 1$, an operator preserves $L^{p_0}(w)$ for any $w \in A_{p_0}$, then necessarily preserves the $L^p(w)$ space for every $1 < p < \infty$ and every $w \in A_p$. Later, in [20] Harboure, Macías and Segovia, proved that the $A_{p,q}$ classes have a similar extrapolation property. Moreover, they also proved that this property is not only exclusive for the boundedness between weighted Lebesgue spaces, but also it is possible to extrapolate based on a continuity behavior of the type $L^s(w) - BMO(w)$ for some $1 < s < \infty$.

Recently in [7], the authors demonstrate extrapolation results that allow us to obtain continuity properties of certain operators of the type $L^p(w^p) - L^q(w^q)$ or $L^p(w^p) - \mathcal{L}_\delta(w)$ starting with hypothesis of continuity of the type $L^s(w^s) - \mathcal{L}_\delta(w)$ for some related parameters. Here, $\mathcal{L}_\delta(w)$, $0 < \delta < 1$, denotes the weighted integral Lipschitz space, i.e., the set of locally integrable function f such that

$$\frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta}{n}}} \int_B |f(x) - f_B| dx \leq C$$

holds for every ball $B \subset \mathbb{R}^n$.

Our first objective is to extend some of the results seen in [7] to $\mathcal{L}_\delta(w)$, an appropriate weighted integral Lipschitz space adapted to the Schrödinger context.

We are also interested in establishing extrapolation results of the type described above in the variable exponent spaces context. In this line we developed extrapolation results starting from hypothesis which involves inequalities of the type $L^s(w^s) - \mathcal{L}_\delta(w)$, and obtaining weighted estimates of the type $L^{p(\cdot)} - L^{q(\cdot)}$ or $L^{p(\cdot)} - \mathcal{L}_{\delta(\cdot)}$, where the last space is a variable version of the space $\mathcal{L}_\delta(w)$.

The structure of this paper is the following. In Section 2 we give some preliminaries and state our main theorems. In Section 3 we state and prove the auxiliary results which are important tools in order to prove the theorems stated in Section 2. Later, in Section 4 we deal with the proofs of the main results. Finally, in Section 5, we give some applications to the variable exponent spaces context by mean of extrapolation techniques.

Throughout this paper, unless otherwise indicated, we will use C and c to denote constants, which are not necessarily the same at each occurrence. We will say that $A \lesssim B$ when there exists a constant $c > 0$ such that $A \leq cB$ and we will write $A \simeq B$ whenever $A \lesssim B$ and $B \lesssim A$.

2. PRELIMINARIES AND MAIN RESULTS

From now on, we call a critical radius function to any positive function ρ that satisfies (1.1). Clearly, if ρ is such a function, so it is $\beta\rho$ for any $\beta > 0$.

Let's denote by \mathcal{B}_ρ the family of subcritical balls of \mathbb{R}^n , i.e., the set of balls $B(x, r)$ with $x \in \mathbb{R}^n$ and $r \leq \rho(x)$.

Taking into account what has been done in [3] (see also [6] and [27]) for the case of a Schrödinger operator, we will consider classes of weights given in terms of a critical radius function.

By a weight w we mean a locally integrable function such that $0 < w(x) < \infty$ a.e. We say that the weight w belongs to the A_p^ρ class, for $1 < p < \infty$, if there exist constants $\theta \geq 0$ and $C > 0$ such that the inequality

$$\left(\frac{1}{|B|} \int_B w \, dy\right)^{1/p} \left(\frac{1}{|B|} \int_B w^{-\frac{1}{p-1}} \, dy\right)^{1/p'} \leq C \left(1 + \frac{r}{\rho(x)}\right)^\theta \tag{2.1}$$

holds for every ball $B = B(x, r) \subset \mathbb{R}^n$. For the case $p = 1$, we will say that w belongs to the class A_1^ρ if there are constants $\theta \geq 0$ and $C > 0$ such that the inequality

$$\frac{1}{|B|} \int_B w \, dy \leq C \left(\inf_{x \in B} w(x)\right) \left(1 + \frac{r}{\rho(x)}\right)^\theta$$

holds for every ball $B = B(x, r) \subset \mathbb{R}^n$.

On the other hand, we will say that a weight w belongs to the class $A_{p,q}^\rho$, for $p, q \in (1, \infty)$, if there exist constants $\theta \geq 0$ and $C > 0$ such that the inequality

$$\left(\frac{1}{|B|} \int_B w^q \, dy\right)^{1/q} \left(\frac{1}{|B|} \int_B w^{-p'} \, dy\right)^{1/p'} \leq C \left(1 + \frac{r}{\rho(x)}\right)^\theta \tag{2.2}$$

holds for every ball $B = B(x, r) \subset \mathbb{R}^n$. In the limiting case $q = \infty$, we will say that $w \in A_{p,\infty}^\rho$, if there exist $\theta > 0$ and $C > 0$ such that the inequality

$$\|w\chi_B\|_\infty \left(\frac{1}{|B|} \int_B w^{-p'} \, dy\right)^{1/p'} \leq C \left(1 + \frac{r}{\rho(x)}\right)^\theta$$

holds for every ball $B = B(x, r) \subset \mathbb{R}^n$, where $\|w\chi_B\|_\infty = \sup_{x \in B} w(x)$.

We will also consider two other classes of weights, namely $A_{p,q}^{\rho, \text{loc}}$ and $A_{p,q}^{\rho, \text{loc}}$, which are formed by those weights w that satisfy (2.1) and (2.2), respectively, only for the balls in \mathcal{B}_ρ . From their definitions it is clear that $A_p^\rho \subset A_p^{\rho, \text{loc}}$ and $A_{p,q}^\rho \subset A_{p,q}^{\rho, \text{loc}}$, for any $1 < p < \infty$ and $1 < q \leq \infty$.

Some properties of these classes of weights that can be derived from their definition and that will be useful to us are the following.

Proposition 2.1. *Let $1 \leq p \leq q \leq \infty$. The following are satisfied:*

- (1) *If $w \in A_p^\rho$, then $w \in A_q^\rho$.*
- (2) *If $w_1, w_2 \in A_1^\rho$, then $w_1 w_2^{1-p} \in A_p^\rho$.*
- (3) *If $p > 1$, $w \in A_{p,q}^\rho$ if and only if $w^{-p'} \in A_{1+\frac{p'}{q}}^\rho$.*
- (4) *If $p > 1$ and $q < \infty$, $w \in A_{p,q}^\rho$ if and only if $w^q \in A_{1+\frac{q}{p}}^\rho$.*
- (5) *If $w \in A_{p,q}^\rho$ and $p > 1$, then $w \in A_{p,r}^\rho$, for $q \geq r \geq p$.*
- (6) *If $w \in A_{p,q}^\rho$ and $p > 1$, then $w \in A_{r,q}^\rho$, for $r \geq p$.*

Recall that given a weight w , $L^p(w)$ denotes the space of functions f such that

$$\int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx < \infty.$$

The classes A_p^ρ are deeply connected to the family of maximal operators M_ρ^θ , defined for $\theta > 0$ by means of

$$M_\rho^\theta f(x) = \sup_{r>0} \left(1 + \frac{r}{\rho(x)}\right)^{-\theta} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

Indeed, these are bounded in $L^p(w)$, for $1 < p < \infty$, provided that $w \in A_p^\rho$, and of weak type $(1, 1)$ with respect to w , for $w \in A_1^\rho$ (see Proposition 3 in [1] and Proposition 4.2 in [5]).

As stated above we are interested in introducing Lipschitz type integral spaces in the Schrödinger context and establishing certain extrapolation results involving them.

In the following, we denote by f_B the average of f over the B ball, i.e., $\frac{1}{|B|} \int_B f$.

Given a weight w , we define the space $\mathcal{L}_\delta(w)$, for $0 \leq \delta < 1$, as the set of all functions f on $L^1_{loc}(\mathbb{R}^n)$ such that

$$\frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta}{n}}} \int_B |f(y) - f_B| dy \leq C, \quad \forall B \in \mathcal{B}_\rho. \tag{2.3}$$

and

$$\frac{\|w\chi_{B(x,\rho(x))}\|_\infty}{|B(x,\rho(x))|^{1+\frac{\delta}{n}}} \int_{B(x,\rho(x))} |f(y)| dy \leq C, \quad \forall x \in \mathbb{R}^n. \tag{2.4}$$

We can consider a norm in $\mathcal{L}_\delta(w)$ as the smallest constant that satisfies (2.3) and (2.4) simultaneously, and denote it by $\|f\|_{\mathcal{L}_\delta(w)}$.

Remark 2.2. A function f that satisfies the inequality (2.4) for a ball B , also satisfies (2.3) for the same ball. This essentially follows from the fact that $\int_B |f_B| \leq \int_B |f(x)|$.

Remark 2.3. When $\delta = 0$, the space $\mathcal{L}_0(w)$ coincides with the space $BMO_\rho(w)$ originally considered in [1].

Although our goal is to use extrapolation to prove the boundedness of some specific operators, we will state our results in a more abstract way. Following the approach established in [11] (see also [14] and [10]) we will present our extrapolation theorems for pairs of measurable, nonnegative functions (f, g) belonging to some family \mathcal{F} . Henceforth, if we write

$$\|f\|_X \leq \|g\|_Y, \quad (f, g) \in \mathcal{F},$$

where X and Y are spaces of functions (i.e., weighted Lebesgue spaces, classical or variable), then we mean that this inequality is true for any pair $(f, g) \in \mathcal{F}$ such that the left-hand side of this inequality is finite.

We are now in a position to state our first extrapolation results taking as hypotheses inequalities of the type $L^s(w^s) - \mathcal{L}_\delta(w)$. The first one gives $L^p(w^p) - L^q(w^q)$ type estimates and the second result gives $L^p(w^p) - \mathcal{L}_{\tilde{\delta}}(w)$ estimates with $0 \leq \delta \leq \tilde{\delta}$.

Theorem 2.4. *Let $0 \leq \delta < 1$, $1 < \beta < \frac{n}{\delta}$ and $\frac{1}{s} = \frac{1}{\beta} - \frac{\delta}{n}$ be such that the inequality*

$$\|f\|_{\mathcal{L}_\delta(w)} \leq C\|gw\|_s, \quad (f, g) \in \mathcal{F},$$

is satisfied for all $w \in A_{s,\infty}^p$ and some constant $C > 0$. Then, there exists a constant $c > 0$ such that the inequality

$$\|fw\|_q \leq C\|gw\|_p, \quad (f, g) \in \mathcal{F},$$

is satisfied for any p and q such that $1 < p < \beta$, $\frac{1}{\beta} = \frac{1}{p} - \frac{1}{q}$ and any weight $w \in A_{p,q}^p$.

Theorem 2.5. *Let $0 \leq \delta < 1$, $1 < \beta < \frac{n}{\delta}$ and $\frac{1}{s} = \frac{1}{\beta} - \frac{\delta}{n}$ be such that the inequality*

$$\|f\|_{\mathcal{L}_\delta(w)} \leq C\|gw\|_s, \quad (f, g) \in \mathcal{F},$$

is satisfied for all $w \in A_{s,\infty}^p$ and some constant $C > 0$. Then, there exists a constant $c > 0$ such that the inequality

$$\|f\|_{\mathcal{L}_\delta(w)} \leq C\|gw\|_p, \quad (f, g) \in \mathcal{F},$$

is true for any $0 \leq \tilde{\delta} \leq \delta$, $\frac{1}{p} = \frac{1}{\beta} - \frac{\tilde{\delta}}{n}$ and every weight $w \in A_{p,\infty}^p$.

We are also interested in obtaining extrapolation results of the type described above in the variable Lebesgue space context. We begin with some definitions and notations related to these spaces.

Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty]$ be a measurable function. Given a measurable set $A \subset \mathbb{R}^n$ we define

$$p^-(A) := \operatorname{ess\,inf}_{x \in A} p(x), \quad p^+(A) := \operatorname{ess\,sup}_{x \in A} p(x).$$

For simplicity we denote $p^- = p^-(\mathbb{R}^n)$ and $p^+ = p^+(\mathbb{R}^n)$.

Given $p(\cdot)$, the conjugate exponent $p'(\cdot)$ is defined pointwise

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

where we let $p'(x) = \infty$ if $p(x) = 1$.

By $\mathcal{P}(\mathbb{R}^n)$ we will designate the collection of all measurable functions $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty]$ and by $\mathcal{P}^*(\mathbb{R}^n)$ the set of $p \in \mathcal{P}(\mathbb{R}^n)$ such that $p^+ < \infty$.

Given $p \in \mathcal{P}^*(\mathbb{R}^n)$, we say that a measurable function f belongs to $L^{p(\cdot)}(\mathbb{R}^n)$ if for some $\lambda > 0$, the modular of f/λ associated with $p(\cdot)$, that is,

$$\varrho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx,$$

is finite. A Luxemburg type norm can be defined in $L^{p(\cdot)}(\mathbb{R}^n)$ by taking

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \leq 1\}.$$

These spaces are special cases of Musieliaik-Orlicz spaces (see [24]), and generalize the classical Lebesgue spaces. For more information see, for example [13, 16, 21].

We will denote with $L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n)$ the space of functions f such that $f\chi_B \in L^{p(\cdot)}(\mathbb{R}^n)$ for every ball $B \subset \mathbb{R}^n$.

In the case of constant exponents, the L^p norm and the modular differ only by one exponent. In variable Lebesgue spaces their relation is less direct, as shown by the following result, whose proof can be found in [13, Corollary 2.23].

Proposition 2.6. *Let $p(\cdot) \in \mathcal{P}^*(\mathbb{R}^n)$. Hence*

- (1) *if $\varrho_{p(\cdot)}(f) \leq C$, then $\|f\|_{p(\cdot)} \leq \max\{C^{1/p^-}, C^{1/p^+}\}$;*
- (2) *if $\|f\|_{p(\cdot)} \leq C$, then $\varrho_{p(\cdot)}(f) \leq \max\{C^{p^+}, C^{p^-}\}$.*

In the classical L^p spaces, $1 < p < \infty$, the norm can be characterized using the identity

$$\|f\|_p = \sup \int_{\mathbb{R}^n} f(x)g(x) dx,$$

where the supremum is considered over all functions g such that $g \in L^{p'}$ and $\|g\|_{p'} \leq 1$. Analogously, we have the following result for variable Lebesgue spaces.

Proposition 2.7 ([13, Theorem 2.34]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, f a measurable function and*

$$\| \|f\| \|_{p(\cdot)} = \sup \left\{ \int_{\mathbb{R}^n} f(x)g(x) dx : \|g\|_{p'(\cdot)} \leq 1 \right\}.$$

Then,

$$c \| \|f\| \|_{p(\cdot)} \leq \|f\|_{p(\cdot)} \leq C \| \|f\| \|_{p(\cdot)},$$

where the constants c and C depend only on $p(\cdot)$.

On the other hand, analogously to the previous case, Hölder’s inequality is also valid for variable exponents but with a constant on the right-hand side of it.

Proposition 2.8 ([16, Lema 3.2.20]). *Given $s(\cdot), p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, be such that $1/s(\cdot) = 1/p(\cdot) + 1/q(\cdot)$. Then, for $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$*

$$\|fg\|_{s(\cdot)} \leq 2 \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}.$$

Moreover, if $s(\cdot) \equiv 1$, the inequality above gives

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq 2 \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$

Another elementary but useful property of the classical Lebesgue norm is that it is homogeneous in the exponent, more precisely $\|f^s\|_p = \|f\|_{sp}^s$ for $1 < s < \infty$ and non-negative f . This property also extends to variable Lebesgue spaces as follows.

Proposition 2.9 ([13, Proposition 2.18]). *Let $p(\cdot) \in \mathcal{P}^*(\mathbb{R}^n)$, so for all $s, 1/p^- \leq s < \infty$,*

$$\| |f|^s \|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s.$$

The following conditions on the exponent arise in connection with the boundedness of the Hardy–Littlewood maximal operator M in $L^{p(\cdot)}(\mathbb{R}^n)$ (see, for example, [15], [13] or [16]). We will say that p is log-Hölder continuous, and we will write $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, if $p \in \mathcal{P}^*(\mathbb{R}^n)$ and if there are constants $C > 0$ and p_∞ such that

$$|p(x) - p(y)| \leq -\frac{C}{\log(|x - y|)}, \quad x, y \in \mathbb{R}^n, |x - y| < 1/2,$$

and

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)}, \quad x \in \mathbb{R}^n.$$

The statement below establishes a reverse Hölder property under certain conditions on the exponents involved.

Proposition 2.10 ([23, Lemma 2.7]). *Let $p(\cdot), r(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ such that $r(\cdot) \leq p(\cdot)$. Suppose that $1/r(\cdot) = 1/p(\cdot) + 1/s(\cdot)$. Then, for every ball $B \subset \mathbb{R}^n$*

$$\|\chi_B\|_{r(\cdot)} \simeq \|\chi_B\|_{p(\cdot)} \|\chi_B\|_{s(\cdot)}. \tag{2.5}$$

Remark 2.11. It is straightforward to see that inequality (2.5) can be also written as

$$\|\chi_B\|_{p'(\cdot)} \simeq \|\chi_B\|_{r'(\cdot)} \|\chi_B\|_{s(\cdot)}.$$

Given a weight w and $p \in \mathcal{P}(\mathbb{R}^n)$, we define the weighted variable Lebesgue space $L^{p(\cdot)}(w)$ to be the set of all measurable functions f such that $fw \in L^{p(\cdot)}$, and we write

$$\|f\|_{L^{p(\cdot)}(w)} = \|f\|_{p(\cdot),w} = \|fw\|_{p(\cdot)}.$$

Thus, we say that an operator T is bounded on $L^{p(\cdot)}(w)$ if

$$\|Tfw\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)},$$

for all $f \in L^{p(\cdot)}(w)$.

Following [6] we define classes of variable weights associated with a function of critical radius ρ . Given a critical radius function ρ and $p \in \mathcal{P}(\mathbb{R}^n)$ we introduce the $A_{p(\cdot)}^{\rho, \text{loc}}$ the class of weights as those w for which there exists a constant $C > 0$ such that the inequality

$$\|w\chi_B\|_{p(\cdot)} \|w^{-1}\chi_B\|_{p'(\cdot)} \leq C|B|$$

holds for every ball $B \in \mathcal{B}_\rho$.

On the other hand, we will say that a weight w belongs to the class $A_{p(\cdot)}^\rho$ if there exist constants $\theta \geq 0$ and $C > 0$ such that the inequality

$$\|w\chi_B\|_{p(\cdot)} \|w^{-1}\chi_B\|_{p'(\cdot)} \leq C|B| \left(1 + \frac{r}{\rho(x)}\right)^\theta$$

holds for all balls $B = B(x, r) \subset \mathbb{R}^n$.

The following result shows the connection between the weights $A_{p(\cdot)}^\rho$ and the boundedness of the operators M_ρ^θ in $L^{p(\cdot)}(w)$.

Theorem 2.12 ([6, Theorem 5]). *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^- > 1$. Then, a weight $w \in A_{p(\cdot)}^\rho$ if and only if there exists $\theta > 0$ such that M_ρ^θ is bounded on $L^{p(\cdot)}(w)$.*

Remark 2.13. It follows from the above definition that if $w \in A_{p(\cdot)}^\rho$, then $w^{-1} \in A_{p'(\cdot)}^\rho$.

Our second pair of results yields off-diagonal inequalities between two different weighted variable Lebesgue spaces. In the case of the constant exponent, the first of these was first demonstrated in [20], while for the variable case, the same was done in [12] and [10] with and without weights, respectively.

To establish such a result we first define an appropriate class of weights that generalizes to the weights $A_{p,q}^\rho$.

Given a critical radius function ρ and $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, we say that a weight w belongs to class $A_{p(\cdot),q(\cdot)}^\rho$ if there exists a positive constant C such that

$$\|w\chi_B\|_{q(\cdot)}\|w^{-1}\chi_B\|_{p'(\cdot)} \leq C\|\chi_B\|_{q(\cdot)}\|\chi_B\|_{p'(\cdot)}\left(1 + \frac{r}{\rho(x)}\right)^\theta, \tag{2.6}$$

for some $\theta > 0$ and any ball $B = B(x, r)$ in \mathbb{R}^n . When $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $p(\cdot) = q(\cdot)$ this is the $A_{p(\cdot)}^\rho$ class.

Remark 2.14. If for some constant $0 < \gamma < 1$, $\frac{1}{p(x)} - \frac{1}{q(x)} = \gamma$, the condition (2.6) is equivalent to the following

$$\|w\chi_B\|_{q(\cdot)}\|w^{-1}\chi_B\|_{p'(\cdot)} \leq C|B|^{1-\gamma}\left(1 + \frac{r}{\rho(x)}\right)^\theta.$$

This definition of class $A_{p(\cdot),q(\cdot)}$ was adopted in [12] in the classical case, i.e., $V = 0$.

We also say that $w \in A_{p(\cdot),\infty}^\rho$ if there exists a positive constant C such that

$$\|w\chi_B\|_\infty\|w^{-1}\chi_B\|_{p'(\cdot)} \leq C\|\chi_B\|_{p'(\cdot)}\left(1 + \frac{r}{\rho(x)}\right)^\theta,$$

for some $\theta > 0$ and any ball $B = B(x, r)$ in \mathbb{R}^n .

We are now in a position to establish another main outcome of this section.

Theorem 2.15. *Let p_0 and q_0 be such that $1 < p_0 < q_0 < \infty$ and suppose that, for all $w_0 \in A_{p_0,q_0}^\rho$,*

$$\|fw_0\|_{q_0} \leq C\|gw_0\|_{p_0}, \quad (f, g) \in \mathcal{F}. \tag{2.7}$$

Be also $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{\sigma'}.$$

Then, if $w \in A_{p(\cdot),q(\cdot)}^\rho$ and $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $q^- > \sigma$, it follows that

$$\|fw\|_{q(\cdot)} \leq C\|gw\|_{p(\cdot)}, \quad (f, g) \in \mathcal{F}.$$

As a direct consequence of Theorems 2.4 and 2.15 we obtain the following extrapolation result.

Theorem 2.16. *Let $1 < \beta < \frac{n}{\delta}$, $0 \leq \delta < 1$ and $\frac{1}{s} = \frac{1}{\beta} - \frac{\delta}{n}$ be such that the inequality*

$$\|f\|_{\mathcal{L}_\delta(w)} \leq C\|gw\|_s, \quad (f, g) \in \mathcal{F},$$

is satisfied for all $w \in A_{s,\infty}^p$ and some constant $C > 0$. Be also $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{\beta}.$$

Then, if $w \in A_{p(\cdot),q(\cdot)}^p$ and $q(\cdot) \in \mathcal{P}^{\text{log}}(\mathbb{R}^n)$ with $q^- > \beta'$, it follows that

$$\|fw\|_{q(\cdot)} \leq C\|gw\|_{p(\cdot)}, \quad (f, g) \in \mathcal{F}.$$

To establish our next result, we will consider a variable version of the space $\mathcal{L}_\delta(w)$.

Let w be a weight, $\gamma > 1$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $\gamma \leq p^- \leq p(x) \leq p^+ < \frac{n\gamma}{(n-\gamma)_+}$ and let $\frac{\delta(x)}{n} = \frac{1}{\gamma} + \frac{1}{p(x)}$ (by β_+ we understand β if $\beta > 0$ and 0 is $\beta \leq 0$). The space $\mathcal{L}_\delta(\cdot)(w)$ is defined by the set of all functions f on $L_{\text{loc}}^1(\mathbb{R}^n)$ such that

$$\frac{\|w\chi_B\|_\infty}{|B|^{\frac{1}{\gamma}}\|\chi_B\|_{p'(\cdot)}} \int_B |f(y) - f_B| dy \leq C, \quad \forall B \in \mathcal{B}_\rho. \tag{2.8}$$

and

$$\frac{\|w\chi_{B(x,\rho(x))}\|_\infty}{|B(x,\rho(x))|^{\frac{1}{\gamma}}\|\chi_{B(x,\rho(x))}\|_{p'(\cdot)}} \int_{B(x,\rho(x))} |f(y)| dy \leq C, \quad \forall x \in \mathbb{R}^n. \tag{2.9}$$

When $p(\cdot)$ is equal to a constant p , this space coincides with the space $\mathcal{L}_{n/\gamma-n/p}(w)$ defined above.

We denote by $\|f\|_{\mathcal{L}_\delta(\cdot)(w)}$ the smallest constant that satisfies (2.8) and (2.9) simultaneously.

We are now in a position to state the last result of our interest.

Theorem 2.17. *Let $\gamma > 1$, $0 \leq \delta < \min\{n/\gamma, 1\}$ and $s > 1$ be such that $\frac{1}{s} = \frac{1}{\gamma} - \frac{\delta}{n}$. Let $p(\cdot) \in \mathcal{P}^{\text{log}}(\mathbb{R}^n)$ with $p^- > 1$, and $0 \leq \tilde{\delta}(\cdot) < \delta$ such that $\frac{1}{p(\cdot)} = \frac{1}{\gamma} - \frac{\tilde{\delta}(\cdot)}{n}$. Let us suppose that for every $w^\eta \in A_{\frac{s}{\eta},\infty}^p$, where $1 < \eta < p^-$,*

$$\|f\|_{\mathcal{L}_\delta(w)} \leq C\|gw\|_s, \quad (f, g) \in \mathcal{F},$$

Then, for every $w^\eta \in A_{\frac{p(\cdot)}{\eta},\infty}^p$ it holds that

$$\|f\|_{\mathcal{L}_{\tilde{\delta}(\cdot)}(w)} \leq C\|gw\|_{p(\cdot)}, \quad (f, g) \in \mathcal{F}.$$

3. AUXILIARY RESULTS

Before proving the main results, we give several technical results necessary for proof. The first is a version of Rubio de Francia algorithm.

Lemma 3.1. *Let $v \in A_m^p$ with $m \geq 1$. Then, for any $h \geq 0$ belonging to $L^{m'}(v)$, there exists $H \in L^{m'}(v)$ such that*

- (1) $h(x) \leq H(x)$ a.e. $x \in \mathbb{R}^n$.
- (2) $\|H\|_{L^{m'}(v)} \leq C\|h\|_{L^{m'}(v)}$.
- (3) $Hv \in A_1^p$.

Proof. Following the so-called Rubio de Francia algorithm, it is sufficient to consider $H = \mathcal{R}h$ defined by

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{[(M_{\rho}^{\theta})']^k h(x)}{2^k \|(M_{\rho}^{\theta})'\|_{L^{m'}(v)}^k},$$

where $(M_{\rho}^{\theta})'f = M_{\rho}^{\theta}(fv)/v$ and $[(M_{\rho}^{\theta})']^k$, for $k \geq 1$ is k times the composition of the operator $(M_{\rho}^{\theta})'$ and $[(M_{\rho}^{\theta})']^0$ denotes the identity operator.

Hence, the properties of H follow easily from its definition and from bounding properties of M_{ρ}^{θ} . □

We now define a localized version of the maximal sharp function as follows.

Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we define maximal sharp local function as

$$f_{\delta}^{\sharp, \rho}(x) = \sup_{B \in \mathcal{B}_{\rho}} \frac{\chi_B(x)}{|B|^{1+\frac{\delta}{n}}} \int_B |f(y) - f_B| dy + \sup_{B=B(z, \rho(z))} \frac{\chi_B(x)}{|B|^{1+\frac{\delta}{n}}} \int_B |f(y)| dy.$$

As expected, the space $\mathcal{L}_{\delta}(w)$ can be described by the operator $f_{\delta}^{\sharp, \rho}$ defined above.

Lemma 3.2. *Let $0 \leq \delta < 1$ and w be a weight. Then, there exist positive constants C_1 and C_2 such that*

$$C_1 \|wf_{\delta}^{\sharp, \rho}\|_{\infty} \leq \|f\|_{\mathcal{L}_{\delta}(w)} \leq C_2 \|wf_{\delta}^{\sharp, \rho}\|_{\infty}.$$

Proof. The proof of the lemma follows in a similar way to that given in Lemma 2 of [1] for the case $\delta = 0$. □

The following result is used in the proof of Theorem 2.4. It was proved in Corollary 5 of [1].

Lemma 3.3. *Let $1 < p < \infty$ and $w \in A_s^{\rho, \text{loc}}$ for some $s \geq 1$. If $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ then there exists a constant $C > 0$ such that*

$$\|g\|_{L^p(w)} \leq C \|g_0^{\sharp, \rho}\|_{L^p(w)}.$$

A key element in demonstrating our extrapolation results is the following generalization of a result given by Calderón and Scott in [9, Proposition 4.6].

Proposition 3.4. *Let $0 \leq \tilde{\delta} < \delta < 1$, $1 < p \leq \frac{n}{\delta - \tilde{\delta}}$ $y \frac{1}{r} = \frac{1}{p} - \frac{\delta - \tilde{\delta}}{n}$. If $w \in A_{p,r}^{\rho, \text{loc}}$ then there exists a positive constant C such that*

$$\|f_{\tilde{\delta}}^{\sharp, \rho} w\|_r \leq C \|f_{\delta}^{\sharp, \rho} w\|_p.$$

The proof of Proposition 3.4 is a consequence of the following lemma which involves a localized fractional maximal function.

Given a locally integrable function f and $0 \leq \delta < 1$, $M_{\delta}^{\rho, \text{loc}}$ is defined as follows,

$$M_{\delta}^{\rho, \text{loc}} f(x) = \sup_{B \in \mathcal{B}_{\rho}} \frac{\chi_B(x)}{|B|^{1-\frac{\delta}{n}}} \int_B |f(y)| dy.$$

Lemma 3.5. *Let $0 \leq \tilde{\delta} \leq \delta < 1$. Then,*

$$f_{\tilde{\delta}}^{\sharp, \rho}(x) \leq 2M_{\delta - \tilde{\delta}}^{\rho, \text{loc}}(f_{\delta}^{\sharp, \rho})(x).$$

Proof. Let $x \in \mathbb{R}^n$ and $B' \in \mathcal{B}_\rho$ such that $x \in B'$. Thus,

$$\begin{aligned} \frac{1}{|B'|^{1+\frac{\delta}{n}}} \int_{B'} |f(y) - f_{B'}| dy &= \frac{|B'|^{\frac{\delta-\tilde{\delta}}{n}}}{|B'|^{1+\frac{\delta-\tilde{\delta}}{n}}} \int_{B'} |f(y) - f_{B'}| dy \\ &\leq |B'|^{\frac{\delta-\tilde{\delta}}{n}} f_\delta^{\#, \rho}(z), \end{aligned}$$

for any $z \in B'$. Integrating over B' it follows that

$$\begin{aligned} |B'| \frac{1}{|B'|^{1+\frac{\delta}{n}}} \int_{B'} |f(y) - f_{B'}| dy &= \int_{B'} \left(\frac{1}{|B'|^{1+\frac{\delta}{n}}} \int_{B'} |f(y) - f_{B'}| dy \right) dz \\ &\leq |B'|^{\frac{\delta-\tilde{\delta}}{n}} \int_{B'} f_\delta^{\#, \rho}(z) dz, \end{aligned}$$

from which it follows that

$$\begin{aligned} \frac{1}{|B'|^{1+\frac{\delta}{n}}} \int_{B'} |f(y) - f_{B'}| dy &\leq \frac{1}{|B'|^{1-\frac{\delta-\tilde{\delta}}{n}}} \int_{B'} f_\delta^{\#, \rho}(z) dz. \\ &\leq M_{\delta-\tilde{\delta}}^{\rho, \text{loc}}(f_\delta^{\#, \rho})(x). \end{aligned}$$

Thus

$$\sup_{B \in \mathcal{B}_\rho} \frac{\chi_B(x)}{|B|^{1+\frac{\delta}{n}}} \int_B |f(y) - f_B| dy \leq M_{\delta-\tilde{\delta}}^{\rho, \text{loc}}(f_\delta^{\#, \rho})(x). \tag{3.1}$$

In a completely analogous manner, taking $B = B(z, \rho(z))$ with $z \in \mathbb{R}^n$ in place of $B' \in \mathcal{B}_\rho$, it is concluded that

$$\sup_{B=B(z, \rho(z))} \frac{\chi_B(x)}{|B|^{1+\frac{\delta}{n}}} \int_B |f(y)| dy \leq M_{\delta-\tilde{\delta}}^{\rho, \text{loc}}(f_\delta^{\#, \rho})(x). \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$f_\delta^{\#, \rho}(x) \leq 2M_{\delta-\tilde{\delta}}^{\rho, \text{loc}}(f_\delta^{\#, \rho})(x).$$

□

Proof of Proposition 3.4. Suppose that $\|f_\delta^{\#, \rho} w\|_p < \infty$, that is, $f_\delta^{\#, \rho} \in L^p(w^p)$, otherwise, there is nothing to prove.

Let us first consider the case where $p < \frac{n}{\delta-\tilde{\delta}}$ and let r be as in the hypothesis. By the previous lemma we have that

$$\begin{aligned} \|f_\delta^{\#, \rho} w\|_r &= \left(\int_{\mathbb{R}^n} [f_\delta^{\#, \rho} w]^r dx \right)^{1/r} \\ &\leq 2 \left(\int_{\mathbb{R}^d} [M_{\delta-\tilde{\delta}}^{\rho, \text{loc}}(f_\delta^{\#, \rho})]^r w^r dx \right)^{1/r} \\ &= 2 \|M_{\delta-\tilde{\delta}}^{\rho, \text{loc}}(f_\delta^{\#, \rho}) w\|_r. \end{aligned} \tag{3.3}$$

Considering $\delta > \tilde{\delta}$ and $\alpha = \delta - \tilde{\delta} > 0$ we have that $\alpha < n$ and both p and r fulfill the hypotheses of the Theorem 3.1.13 in [8], so that $M_{\delta-\tilde{\delta}}^{\rho, \text{loc}}$ is bounded from $L^p(w^p)$ in $L^r(w^r)$.

Therefore, it follows from (3.3) that

$$\|f_{\tilde{\delta}}^{\#, \rho} w\|_r \leq C \|f_{\delta}^{\#, \rho} w\|_p.$$

On the other hand, if $\delta = \tilde{\delta}$, it follows from the hypothesis that $r = p$. Also, $w \in A_{p,p}^{\rho}$ if and only if $w^p \in A_p^{\rho}$. Then, $w^p \in A_p^{\rho, \text{loc}}$ and from Theorem 1 in [3] it follows that $M_{\delta-\tilde{\delta}}^{\rho, \text{loc}}$ is bounded on $L^p(w^p)$, from which we obtain the result we wanted.

Let us now look at the case where $p = \frac{n}{\delta-\tilde{\delta}}$, so that $r = \infty$. We want to see that if $w \in A_{\frac{n}{\delta-\tilde{\delta}}, \infty}^{\rho, \text{loc}}$, then there exists $C > 0$ such that

$$\|f_{\tilde{\delta}}^{\#, \rho} w\|_{\infty} \leq C \|f_{\delta}^{\#, \rho} w\|_{\frac{n}{\delta-\tilde{\delta}}}.$$

From Lemma 3.5 it follows that

$$\|f_{\tilde{\delta}}^{\#, \rho} w\|_{\infty} \leq 2 \|M_{\delta-\tilde{\delta}}^{\rho, \text{loc}}(f_{\delta}^{\#, \rho})w\|_{\infty}. \tag{3.4}$$

On the other hand, by Proposition 3.1.12 in [8] it follows that if $w^{\frac{n}{\delta-\tilde{\delta}-n}} \in A_1^{\rho, \text{loc}}$, then $M_{\delta-\tilde{\delta}}^{\rho, \text{loc}}$ is bounded from $L^{\frac{n}{\delta-\tilde{\delta}}}(w^{\frac{n}{\delta-\tilde{\delta}}})$ on $L^{\infty}(w)$. It is enough to see that $w^{\frac{n}{\delta-\tilde{\delta}-n}} \in A_1^{\rho, \text{loc}}$. Being $p = \frac{n}{\delta-\tilde{\delta}}$,

$$\frac{n}{\delta - \tilde{\delta} - n} = \frac{1}{\frac{\delta-\tilde{\delta}}{n} - 1} = \frac{1}{\frac{1}{p} - 1} = \frac{1}{-\frac{1}{p'}} = -p',$$

and since $w \in A_{p, \infty}^{\rho, \text{loc}}$ it follows that $w^{-p'} \in A_1^{\rho, \text{loc}}$, which is what we wanted to prove.

Therefore, by Proposition 3.1.12 in [8], we obtain

$$\|M_{\delta-\tilde{\delta}}^{\rho, \text{loc}}(f_{\delta}^{\#, \rho})w\|_{\infty} \leq C \|f_{\delta}^{\#, \rho} w\|_{\frac{n}{\delta-\tilde{\delta}}}. \tag{3.5}$$

Finally, from (3.4) and (3.5) we get that

$$\|f_{\tilde{\delta}}^{\#, \rho} w\|_{\infty} \leq C \|f_{\delta}^{\#, \rho} w\|_{\frac{n}{\delta-\tilde{\delta}}}.$$

□

We will also need some auxiliary results in the variable context.

The following results refer to properties of the weights $A_{p(\cdot), q(\cdot)}^{\rho}$. The proof of the first proposition is straightforward and we omit it.

Proposition 3.6. *Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $w \in A_{p(\cdot), \infty}^{\rho}$ then $w \in A_{p(\cdot), q(\cdot)}^{\rho}$.*

Proposition 3.7. *Let $p(\cdot), q(\cdot) \in \mathcal{P}^{\text{log}}(\mathbb{R}^n)$ such that $q(\cdot) \leq p(\cdot)$. If $w \in A_{q(\cdot), \infty}^{\rho}$ then $w \in A_{p(\cdot), \infty}^{\rho}$.*

Proof. Let $s(\cdot)$ such that $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{s(\cdot)}$. Since $1/p'(\cdot) = 1/q'(\cdot) + 1/s(\cdot)$, from Hölder's inequality, the hypothesis on w and Remark 2.11, we obtain

$$\begin{aligned} \|w\chi_B\|_\infty \|w^{-1}\chi_B\|_{p'(\cdot)} &\leq C \|w\chi_B\|_\infty \|w^{-1}\chi_B\|_{q'(\cdot)} \|\chi_B\|_{s(\cdot)} \\ &\leq C \|\chi_B\|_{q'(\cdot)} \|\chi_B\|_{s(\cdot)} \left(1 + \frac{r}{\rho(x)}\right)^\theta \\ &\leq C \|\chi_B\|_{p'(\cdot)} \left(1 + \frac{r}{\rho(x)}\right)^\theta. \quad \square \end{aligned}$$

Proposition 3.8. *Let $p(\cdot), r(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ such that $p(\cdot) \leq r(\cdot)$, and η satisfying $1 < \eta < p^-$. If $w^\eta \in A_{\frac{p(\cdot)}{\eta}, \infty}^\rho$ then $w \in A_{r(\cdot), \infty}^\rho$.*

Proof. Since $p(\cdot) \leq r(\cdot)$ it follows from Proposition 3.7 that $w^\eta \in A_{\frac{r(\cdot)}{\eta}, \infty}^\rho$. Taking $\beta = 1/(\eta - 1)$ it follows that $\frac{\eta}{r'(\cdot)} = \frac{1}{(r(\cdot)/\eta)'} + \frac{1}{\beta}$.

Then, by Proposition 2.9, the generalized Hölder inequality and Proposition 2.10 we get

$$\begin{aligned} \left(\frac{\|w\chi_B\|_\infty \|w^{-1}\chi_B\|_{r'(\cdot)}}{\|\chi_B\|_{r'(\cdot)}}\right)^\eta &= \frac{\|w^\eta\chi_B\|_\infty \|w^{-\eta}\chi_B\|_{r'(\cdot)/\eta}}{\|\chi_B\|_{r'(\cdot)/\eta}} \\ &\leq C \frac{\|w^\eta\chi_B\|_\infty \|w^{-\eta}\chi_B\|_{(r(\cdot)/\eta)' } \|\chi_B\|_\beta}{\|\chi_B\|_{r'(\cdot)/\eta}} \\ &\leq C \frac{\|w^\eta\chi_B\|_\infty \|w^{-\eta}\chi_B\|_{(r(\cdot)/\eta)' }}{\|\chi_B\|_{(r(\cdot)/\eta)' }} \\ &\leq C \left(1 + \frac{r}{\rho(x)}\right)^\theta, \end{aligned}$$

where we have used $w^\eta \in A_{\frac{r(\cdot)}{\eta}, \infty}^\rho$. We conclude that $w \in A_{r(\cdot), \infty}^\rho$. □

Proposition 3.9. *Let $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $s > 1$ such that $1 < p^- \leq p(x) < q(x) \leq q^+ < \infty$ and $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{1}{s'}$. Then, if $w \in A_{p(\cdot), q(\cdot)}^\rho$ it holds that $w^{-s} \in A_{p'(\cdot)/s}^\rho$.*

Proof. Given a ball $B = B(x, r) \subset \mathbb{R}^n$, by Proposition 2.9, the inequality

$$\|w^{-s}\chi_B\|_{p'(\cdot)/s} \|w^s\chi_B\|_{(p'(\cdot)/s)'} \leq C|B| \left(1 + \frac{r}{\rho(x)}\right)^\theta$$

is equivalent to

$$\|w^{-1}\chi_B\|_{p'(\cdot)} \|w\chi_B\|_{s(p'(\cdot)/s)'} \leq C|B|^{1/s} \left(1 + \frac{r}{\rho(x)}\right)^{\theta/s}.$$

We prove the last one. From Proposition 2.10, since $\frac{1}{s} = \frac{1}{q(\cdot)} + \frac{1}{p'(\cdot)}$, we get that

$$|B|^{1/s} = \|\chi_B\|_s \simeq \|\chi_B\|_{q(\cdot)} \|\chi_B\|_{p'(\cdot)}.$$

Then, by the hypothesis on the weight w and being $q(\cdot) = s(p'(\cdot)/s)'$ we obtain

$$\begin{aligned} \frac{\|w^{-1}\chi_B\|_{p'(\cdot)}\|w\chi_B\|_{s(p'(\cdot)/s)'}}{|B|^{1/s}} &\leq C \frac{\|w\chi_B\|_{q(\cdot)}\|w^{-1}\chi_B\|_{p'(\cdot)}}{\|\chi_B\|_{q(\cdot)}\|\chi_B\|_{p'(\cdot)}} \\ &\leq C \left(1 + \frac{r}{\rho(x)}\right)^\vartheta. \end{aligned}$$

Then, taking $\theta = s\vartheta$, the proof is completed. □

Proposition 3.10. *Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p^- \leq p(x) < q(x) \leq q^+ < \infty$ and we assume that it exists $\sigma > 1$ such that $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{1}{\sigma'}$. Then, $w \in A_{p(\cdot), q(\cdot)}^\rho$ if and only if $w^\sigma \in A_{q(\cdot)/\sigma}^\rho$.*

Remark 3.11. Let us note that being $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{1}{\sigma'}$ we have $\frac{1}{p^-} - \frac{1}{q^-} = \frac{1}{\sigma'}$, that is,

$$\frac{1}{p^-} - 1 = \frac{1}{q^-} - \frac{1}{\sigma},$$

so the condition $p^- > 1$ is equivalent to having $q^- > \sigma$.

Proof. That weight w^σ belongs to $A_{q(\cdot)/\sigma}^\rho$ is equivalent to the existence of constants $\theta > 0$ and $C > 0$ such that

$$\|w^\sigma \chi_B\|_{r(\cdot)}^{1/\sigma} \|w^{-\sigma} \chi_B\|_{r'(\cdot)}^{1/\sigma} \leq C |B|^{1/\sigma} \left(1 + \frac{r}{\rho(x)}\right)^\theta,$$

for any ball $B = B(x, r)$ in \mathbb{R}^n and where $r(x) = q(x)/\sigma$.

Moreover, as $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{\sigma'}$, by virtue of Proposition 2.10 w belongs to $A_{p(\cdot), q(\cdot)}^\rho$ if there are constants $C > 0$ y $\theta' > 0$ such that

$$\begin{aligned} \|w\chi_B\|_{q(\cdot)}\|w^{-1}\chi_B\|_{p'(\cdot)} &\leq C|B|^{1-\frac{1}{\sigma'}} \left(1 + \frac{r}{\rho(x)}\right)^{\theta'} \\ &= C|B|^{1/\sigma} \left(1 + \frac{r}{\rho(x)}\right)^{\theta'}, \end{aligned}$$

for any ball $B = B(x, r)$ in \mathbb{R}^n .

Since $1 < q^- < q^+ < \infty$, we can apply the Proposition 2.9 and obtain that for every ball B ,

$$\|w^\sigma \chi_B\|_{q(\cdot)/\sigma}^{1/\sigma} = \|w\chi_B\|_{q(\cdot)}. \tag{3.6}$$

Now let us look at the relationship that exists between $r'(\cdot)$ and $p'(\cdot)$. On the one hand, we have that

$$r'(x) = \frac{\frac{q(x)}{\sigma}}{\frac{q(x)}{\sigma} - 1} = \frac{q(x)}{q(x) - \sigma}. \tag{3.7}$$

On the other hand,

$$\frac{1}{p'(x)} = 1 - \frac{1}{p(x)} = 1 - \frac{1}{\sigma'} - \frac{1}{q(x)} = \frac{1}{\sigma} - \frac{1}{q(x)},$$

from which it follows that

$$p'(x) = \sigma \frac{q(x)}{q(x) - \sigma} \tag{3.8}$$

Then, from (3.7) and (3.8), it follows that $r'(x) = \frac{1}{\sigma}p'(x)$. Then, again, by Proposition 2.9 and given that $(p')^- \geq 1$ and $p^- > 1$ we get that

$$\|w^{-\sigma}\chi_B\|_{r'(\cdot)}^{1/\sigma} = \|w^{-1}\chi_B\|_{p'(\cdot)}. \tag{3.9}$$

Therefore, from (3.6) and (3.9) it follows that $w \in A_{p(\cdot),q(\cdot)}^\rho$ if, and only if, $w^\sigma \in A_{q(\cdot)/\sigma}^\rho$. □

On the other hand, to construct the weight w_0 necessary to be able to apply the hypothesis of Theorem 2.15, we will use item (2) of Proposition 2.1 and therefore, to find the weights in A_1^ρ , we will consider the following variable version of Rubio de Francia’s extrapolation algorithm.

Proposition 3.12. *Let $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and suppose that v is a weight such that M_ρ^θ is bounded on $L^{r(\cdot)}(v)$, for some $\theta > 0$. For a non-negative function h such that $h \in L^{r(\cdot)}(v)$ we define*

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{(M_\rho^\theta)^k h(x)}{2^k \|M_\rho^\theta\|_{L^{r(\cdot)}(v)}^k}.$$

Then,

- (1) $h(x) \leq \mathcal{R}h(x)$ a.e. $x \in \mathbb{R}^n$.
- (2) $\|\mathcal{R}h\|_{L^{r(\cdot)}(v)} \leq 2\|h\|_{L^{r(\cdot)}(v)}$.
- (3) $\mathcal{R}h \in A_1^\rho$.

Proof. The proof is essentially the same as in the constant exponent case. Property (1) for $\mathcal{R}h$ is immediate, property (2) is deduced from the assumption that M_ρ^θ is bounded in $L^{r(\cdot)}(v)$, and finally, property (3) follows from the fact that M_ρ^θ is sublinear and h is non-negative. □

As a consequence of the previous result we will prove the following corollary which will be used in the proof of Theorem 2.15.

Corollary 3.13. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and σ such that $1 < \sigma < q^-$, $1 < p^- \leq p(x) < q(x) \leq q^+ < \infty$ and $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{1}{\sigma'}$. Also let h_1 and h_2 be non-negative functions such that $h_1 \in L^{q(\cdot)}(w)$ and $h_2 \in L^{(q(\cdot)/\sigma)'}(\mathbb{R}^n)$.*

Then, if $w \in A_{p(\cdot),q(\cdot)}^\rho$ there exist $H_1 \in L^{q(\cdot)}(w)$ and $H_2 \in L^{(q(\cdot)/\sigma)'(\mathbb{R}^n)}$ satisfying

- (1) $h_1(x) \leq H_1(x)$ a.e. $x \in \mathbb{R}^n$.
- (2) $\|H_1\|_{L^{q(\cdot)}(w)} \leq 2\|h_1\|_{L^{q(\cdot)}(w)}$.
- (3) $H_1^\sigma \in A_1^\rho$.
- (1') $h_2(x) \leq H_2(x)$ a.e. $x \in \mathbb{R}^n$.
- (2') $\|H_2\|_{(q(\cdot)/\sigma)'} \leq 2\|h_2\|_{(q(\cdot)/\sigma)'}$.
- (3') $H_2 w^\sigma \in A_1^\rho$.

Proof. Let us begin by proving the existence and properties of H_1 . To do so, let us consider

$$H_1 = (\mathcal{R} h_1^\sigma)^{1/\sigma}.$$

Since $w \in A_{p(\cdot),q(\cdot)}^\rho$, in view of the Proposition 3.10, we have that $w^\sigma \in A_{q(\cdot)/\sigma}^\rho$. Hence, by virtue of Theorem 2.12 and given that $q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $q^- > \sigma$, one has that there exists $\theta > 0$ such that M_ρ^θ is bounded in $L^{q(\cdot)/\sigma}(w^\sigma)$.

Let us also note that since $h_1 \in L^{q(\cdot)}(w)$ it follows by Proposition 2.9 that $h_1^\sigma \in L^{q(\cdot)/\sigma}(w^\sigma)$.

Therefore, applying Proposition 3.12, with $h = h_1^\sigma$, $r(\cdot) = q(\cdot)/\sigma$ and $v = w^\sigma$, we have that

- i) $h_1^\sigma \leq \mathcal{R} h_1^\sigma$ a.e.
- ii) $\|\mathcal{R} h_1^\sigma\|_{L^{q(\cdot)/\sigma}(w^\sigma)} \leq 2\|h_1^\sigma\|_{L^{q(\cdot)/\sigma}(w^\sigma)}$.
- iii) $\mathcal{R} h_1^\sigma \in A_1^\rho$.

It is clear that 1. and 3. follow directly from i) and iii) respectively. On the other hand, 2. is a consequence of ii) and Proposition 2.9, as

$$\begin{aligned} \|H_1 w\|_{q(\cdot)} &= \|(\mathcal{R} h_1^\sigma)^{1/\sigma} (w^\sigma)^{1/\sigma}\|_{q(\cdot)} = \|\mathcal{R} h_1^\sigma w^\sigma\|_{q(\cdot)/\sigma}^{1/\sigma} \\ &\leq 2^{1/\sigma} \|h_1^\sigma w^\sigma\|_{q(\cdot)/\sigma}^{1/\sigma} = 2^{1/\sigma} \|h_1 w\|_{q(\cdot)} \leq 2\|h_1 w\|_{q(\cdot)}. \end{aligned}$$

Let us now look at the existence and properties of H_2 . We will take

$$H_2 = \mathcal{R}(h_2 w^\sigma) w^{-\sigma}.$$

Since $w \in A_{p(\cdot),q(\cdot)}^\rho$, from Proposition 3.10 and the Observation 2.13, we have that $w^{-\sigma} \in A_{(q(\cdot)/\sigma)'}^\rho$. Thus, by Theorem 2.12 and given that $q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $(q^+/\sigma)' = [(q(\cdot)/\sigma)']^- > 1$, one has that there exists $\theta' > 0$ such that $M_\rho^{\theta'}$ is bounded in $L^{(q(\cdot)/\sigma)'}(w^{-\sigma})$.

Let us also note that since $h_2 \in L^{(q(\cdot)/\sigma)'(\mathbb{R}^n)}$ it follows immediately that $h_2 w^\sigma \in L^{(q(\cdot)/\sigma)'(w^{-\sigma})}$.

Therefore, by applying Proposition 3.12, with $h = h_2 w^\sigma$, $r(\cdot) = (q(\cdot)/\sigma)'$ and $v = w^{-\sigma}$, we get

- a) $h_2 w^\sigma \leq \mathcal{R}(h_2 w^\sigma)$ a.e.
- b) $\|\mathcal{R}(h_2 w^\sigma)\|_{L^{(q(\cdot)/\sigma)'(w^{-\sigma})}} \leq 2\|h_2 w^\sigma\|_{L^{(q(\cdot)/\sigma)'(w^{-\sigma})}}$.
- c) $\mathcal{R}(h_2 w^\sigma) \in A_1^\rho$.

Clearly, 1' and 3' are a consequence of a) and c), respectively. Finally, 2. is obtained from b) since

$$\|H_2\|_{(q(\cdot)/\sigma)'} = \|(\mathcal{R} h_2 w^\sigma) w^{-\sigma}\|_{(q(\cdot)/\sigma)'} \leq 2\|(h_2 w^\sigma) w^{-\sigma}\|_{(q(\cdot)/\sigma)'} = 2\|h_2\|_{(q(\cdot)/\sigma)'}. \quad \square$$

In the following corollary, given an exponent function $r(\cdot)$ and $\eta > 1$, we denote by $\bar{r} = \bar{r}(\cdot) = \frac{r(\cdot)}{\eta}$.

Corollary 3.14. *Let $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $s > 1$ such that $1 < p^- \leq p(x) < q(x) \leq q^+ < \infty$ and $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{1}{s}$. Let $1 < \eta < p^-$ and non-negative function $\tilde{h} \in L^{\bar{q}/\bar{s}'}(w^{-\eta \bar{p}' \bar{s}' / \bar{q}})$.*

Then, if $w^\eta \in A_{p(\cdot)/\eta, \infty}^\rho$, there exists $\tilde{H} \in L^{\bar{q}/\bar{s}'}(w^{-\eta \bar{p}' \bar{s}' / \bar{q}})$ such that

- (1) $\tilde{h}(x) \leq \tilde{H}(x)$ a.e. $x \in \mathbb{R}^n$.
- (2) $\|\tilde{H}w^{-\eta\bar{p}'\bar{s}'/\bar{q}}\|_{\bar{q}/\bar{s}'} \leq 2\|\tilde{h}w^{-\eta\bar{p}'\bar{s}'/\bar{q}}\|_{\bar{q}/\bar{s}'}$.
- (3) $\tilde{H}w^{-\eta\bar{p}'} \in A_1^\rho$.

Proof. Since $w^\eta \in A_{p,\infty}^\rho$, it follows from Proposition 3.6 that $w^\eta \in A_{p,\bar{q}}^\rho$. Moreover, given that $\frac{1}{\bar{p}(\cdot)} - \frac{1}{\bar{q}(\cdot)} = \frac{1}{\bar{s}}$, it holds by Proposition 3.9 that $w^{-\eta\bar{s}'} \in A_{\bar{p}'/\bar{s}'}^\rho$.

Thus, the weight $w^{\eta\bar{s}'} \in A_{(\bar{p}'/\bar{s}')'}^\rho = A_{\bar{q}/\bar{s}'}^\rho$ and since that $p \in \mathcal{P}^{\text{log}}(\mathbb{R}^n)$ with $p^- > 1$, it follows that $(\bar{q}/\bar{s}')^- > 1$, and by Theorem 2.12 one has that there exists $\theta > 0$ such that M_ρ^θ is bounded on $L^{\bar{q}/\bar{s}'}(w^{\eta\bar{s}'})$.

In addition, considering that $\tilde{h}w^{-\eta\bar{p}'} \in L^{\bar{q}/\bar{s}'}(w^{\eta\bar{s}'})$, we can apply the Proposition 3.12 con $h = \tilde{h}w^{-\eta\bar{p}'}$, $r(\cdot) = \bar{q}/\bar{s}'$ and $v = w^{\eta\bar{s}'}$. We then have

- i) $\tilde{h}w^{-\eta\bar{p}'} \leq \mathcal{R}(\tilde{h}w^{-\eta\bar{p}'})$, a.e.
- ii) $\|\mathcal{R}(\tilde{h}w^{-\eta\bar{p}'})w^{\eta\bar{s}'}\|_{\bar{q}/\bar{s}'} \leq 2\|\tilde{h}w^{-\eta\bar{p}'}w^{\eta\bar{s}'}\|_{\bar{q}/\bar{s}'}$.
- iii) $\mathcal{R}(\tilde{h}w^{-\eta\bar{p}'}) \in A_1^\rho$.

Thus, if we consider $\tilde{H} = \mathcal{R}(\tilde{h}w^{-\eta\bar{p}'})w^{\eta\bar{p}'}$ it is clear that 1. and 3. follow directly from i) and iii) respectively. Finally, 2. is a consequence of ii) since

$$\begin{aligned} \|\tilde{H}w^{-\eta\bar{p}'\bar{s}'/\bar{q}}\|_{\bar{q}/\bar{s}'} &= \|\mathcal{R}(\tilde{h}w^{-\eta\bar{p}'})w^{\eta\bar{p}'}w^{-\eta\bar{p}'\bar{s}'/\bar{q}}\|_{\bar{q}/\bar{s}'} \\ &= \|\mathcal{R}(\tilde{h}w^{-\eta\bar{p}'})w^{\eta\bar{s}'}\|_{\bar{q}/\bar{s}'} \\ &\leq 2\|\tilde{h}w^{-\eta\bar{p}'}w^{\eta\bar{s}'}\|_{\bar{q}/\bar{s}'} \\ &= 2\|\tilde{h}w^{-\eta\bar{p}'\bar{s}'/\bar{q}}\|_{\bar{q}/\bar{s}'} \end{aligned}$$

□

Similarly to the constant case, the space $\mathcal{L}_{\delta(\cdot)}(w)$ can be described by the operator $f_{\delta(\cdot),s(\cdot),\gamma}^{\sharp,\rho}$ defined for $1 < \gamma < \infty$ and $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $\delta(x)/n = 1/\gamma - 1/s(x)$ by

$$\begin{aligned} f_{\delta(\cdot),s(\cdot),\gamma}^{\sharp,\rho}(x) &= \sup_{B \in \mathcal{B}_\rho} \frac{\chi_B(x)}{|B|^{1/\gamma}\|\chi_B\|_{s'(\cdot)}} \int_B |f(y) - f_B| dy \\ &\quad + \sup_{B=B(z,\rho(z))} \frac{\chi_B(x)}{|B|^{1/\gamma}\|\chi_B\|_{s'(\cdot)}} \int_B |f(y)| dy. \end{aligned}$$

For convenience, in the rest of this paper, we always assume that $f_{\delta(\cdot),s(\cdot)}^{\sharp,\rho}$ denotes $f_{\delta(\cdot),s(\cdot),\gamma}^{\sharp,\rho}$. The proof of the following lemma is analogous to the constant case.

Lemma 3.15. *Let $0 \leq \delta(\cdot) < 1$ and w be a weight. Then, there exist positive constants C_1 and C_2 such that*

$$C_1 \|w f_{\delta(\cdot),s(\cdot)}^{\sharp,\rho}\|_\infty \leq \|f\|_{\mathcal{L}_{\delta(\cdot)}(w)} \leq C_2 \|w f_{\delta(\cdot),s(\cdot)}^{\sharp,\rho}\|_\infty.$$

The following proposition is a key estimate in order to prove the Theorem 2.17. A version in the classical case ($V = 0$) can be found in [25].

Proposition 3.16. *Let $p(\cdot), s(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $\gamma > 1$ such that $\frac{\delta(\cdot)}{n} = \frac{1}{\gamma} - \frac{1}{s(\cdot)}$ and $\frac{\tilde{\delta}(\cdot)}{n} = \frac{1}{\gamma} - \frac{1}{p(\cdot)}$ with $0 \leq \tilde{\delta}(\cdot) < \delta(\cdot) < 1$. Then*

$$\|w f_{\delta(\cdot), p(\cdot)}^{\#, \rho}\|_{\infty} \leq C \|w f_{\delta(\cdot), s(\cdot)}^{\#, \rho}\|_{\frac{n}{\delta(\cdot) - \tilde{\delta}(\cdot)}},$$

for every $w \in A_{n/(\delta(\cdot) - \tilde{\delta}(\cdot)), \infty}^{\rho}$.

Proof. Given $B = B(x_B, r_B) \in \mathcal{B}_{\rho}$, from definition of $f_{\delta(\cdot), s(\cdot)}^{\#, \rho}$, for almost every-where $z \in B$, we have that

$$\frac{1}{|B|^{1/\gamma} \|\chi_B\|_{s'(\cdot)}} \int_B |f(y) - f_B| dy \leq f_{\delta(\cdot), s(\cdot)}^{\#, \rho}(z).$$

By integrating over B and applying Hölder's inequality we obtain

$$\begin{aligned} & \frac{|B|}{|B|^{1/\gamma} \|\chi_B\|_{s'(\cdot)}} \int_B |f(y) - f_B| dy \\ & \leq \int_B f_{\delta(\cdot), s(\cdot)}^{\#, \rho}(z) dz \\ & \leq C \|f_{\delta(\cdot), s(\cdot)}^{\#, \rho} w\|_{\frac{n}{\delta(\cdot) - \tilde{\delta}(\cdot)}} \|w^{-1} \chi_B\|_{\frac{n}{n - (\delta(\cdot) - \tilde{\delta}(\cdot))}}, \end{aligned}$$

from which it follows that

$$\begin{aligned} & \frac{\|w \chi_B\|_{\infty}}{|B|^{1/\gamma} \|\chi_B\|_{p'(\cdot)}} \int_B |f(y) - f_B| dy \\ & \leq C \|f_{\delta(\cdot), s(\cdot)}^{\#, \rho} w\|_{\frac{n}{\delta(\cdot) - \tilde{\delta}(\cdot)}} \|w^{-1} \chi_B\|_{\frac{n}{n - (\delta(\cdot) - \tilde{\delta}(\cdot))}} \frac{\|\chi_B\|_{s'(\cdot)} \|w \chi_B\|_{\infty}}{|B| \|\chi_B\|_{p'(\cdot)}}. \end{aligned}$$

Considering the hypothesis on weight w and since $r_B \leq \rho(x_B)$, we get that

$$\begin{aligned} & \|w^{-1} \chi_B\|_{\frac{n}{n - (\delta(\cdot) - \tilde{\delta}(\cdot))}} \frac{\|\chi_B\|_{s'(\cdot)} \|w \chi_B\|_{\infty}}{|B| \|\chi_B\|_{p'(\cdot)}} \\ & \leq C \|\chi_B\|_{\frac{n}{n - (\delta(\cdot) - \tilde{\delta}(\cdot))}} \frac{\|\chi_B\|_{s'(\cdot)}}{|B| \|\chi_B\|_{p'(\cdot)}} \left(1 + \frac{r_B}{\rho(x_B)}\right)^{\theta} \\ & \leq C \|\chi_B\|_{\frac{n}{n - (\delta(\cdot) - \tilde{\delta}(\cdot))}} \frac{\|\chi_B\|_{s'(\cdot)}}{|B| \|\chi_B\|_{p'(\cdot)}}. \end{aligned}$$

Later, since $\frac{n - (\delta(\cdot) - \tilde{\delta}(\cdot))}{n} = \frac{1}{s(\cdot)} + \frac{1}{p'(\cdot)}$, from Proposition 2.10 we have that

$$\begin{aligned} & \|w^{-1} \chi_B\|_{\frac{n}{n - (\delta(\cdot) - \tilde{\delta}(\cdot))}} \frac{\|\chi_B\|_{s'(\cdot)} \|w \chi_B\|_{\infty}}{|B| \|\chi_B\|_{p'(\cdot)}} \leq C \|\chi_B\|_{s(\cdot)} \|\chi_B\|_{p'(\cdot)} \frac{\|\chi_B\|_{s'(\cdot)}}{|B| \|\chi_B\|_{p'(\cdot)}} \\ & \leq C \frac{\|\chi_B\|_{s(\cdot)} \|\chi_B\|_{s'(\cdot)}}{|B|} \\ & \leq C. \end{aligned}$$

Thus,

$$\frac{\|w\chi_B\|_\infty}{|B|^{1/\gamma}\|\chi_B\|_{p'(\cdot)}} \int_B |f(y) - f_B| dy \leq C \|f_{\delta(\cdot),s(\cdot)}^{\sharp,\rho} w\|_{\frac{n}{\delta(\cdot)-\delta(\cdot)}}. \tag{3.10}$$

On the other hand, considering $B' = B(t, \rho(t))$, we get

$$\frac{1}{|B'|^{1/\gamma}\|\chi_{B'}\|_{s'(\cdot)}} \int_{B'} |f(y)| dy \leq f_{\delta(\cdot),s(\cdot)}^{\sharp,\rho}(z),$$

for a.e. $z \in B'$. From here, proceeding analogously to the above, we get that

$$\frac{\|w\chi_{B'}\|_\infty}{|B'|^{1/\gamma}\|\chi_{B'}\|_{p'(\cdot)}} \int_{B'} |f(y)| dy \leq C \|f_{\delta(\cdot),s(\cdot)}^{\sharp,\rho} w\|_{\frac{n}{\delta(\cdot)-\delta(\cdot)}}. \tag{3.11}$$

Finally, from (3.10) and (3.11), the definition of $\|f\|_{\mathcal{L}_{\delta(\cdot)}(w)}$ and Lemma 3.15, the thesis follows. \square

4. PROOF OF THE MAIN RESULTS

In this section we give the proofs of our main theorems. We begin with the proof of Theorem 2.4.

Proof of Theorem 2.4. Let $w \in A_{p,q}^\rho$ and $f \in L^q(w^q)$. Without loss of generality, we can assume that $\|gw\|_p < \infty$, since otherwise there is nothing to prove, and we can also assume that $\|gw\|_p > 0$, since otherwise $g(x) = 0$ at almost every point and then, from the hypothesis, we would also have $f(x) = 0$ at almost every point.

Let us consider $h(x) = \left(\frac{|g(x)|w(x)^{p'}}{\|gw\|_p}\right)^{p-s}$, $\tilde{h} = h^{-s'/s}$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{s}$. Hence, $\tilde{h} \in L^{r/s'}(w^{-p'})$ and

$$\int_{\mathbb{R}^n} \tilde{h}^{r/s'} w^{-p'} dx = 1.$$

Since $w \in A_{p,q}^\rho$ and $\frac{1}{r} = \frac{1}{\beta} + \frac{1}{q} - \frac{1}{s} = \frac{\delta}{n} + \frac{1}{q}$, it follows that $q \geq r$ and so $w \in A_{p,r}^\rho$. Therefore, by Proposition 2.1, $w^{-p'} \in A_{1+\frac{p'}{r}}^\rho$ and applying Lemma 3.1, with $m = 1 + \frac{p'}{r}$ and $v = w^{-p'}$, we know it exists $\tilde{H} \in L^{r/s'}(w^{-p'})$ such that $\tilde{H} \geq \tilde{h}$ and

$$\begin{aligned} \|\tilde{H}^{1/s'} w^{-p'/r}\|_r^r &= \int_{\mathbb{R}^n} \tilde{H}^{r/s'} w^{-p'} dx \\ &\leq C \int_{\mathbb{R}^n} \tilde{h}^{r/s'} w^{-p'} dx \\ &= C. \end{aligned} \tag{4.1}$$

Hence,

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |g|^p w^p dx\right)^{1/p} &= \left(\int_{\mathbb{R}^n} |gw^{p'}|^s h w^{-p'} dx\right)^{1/s} \\ &= \left(\int_{\mathbb{R}^n} |gw^{p'}|^s (\tilde{h}^{-1/s'})^s w^{-p'} dx\right)^{1/s} \\ &\geq \left(\int_{\mathbb{R}^n} |gw^{p'}|^s (\tilde{H}^{-1/s'})^s w^{-p'} dx\right)^{1/s} \\ &= \left(\int_{\mathbb{R}^n} |g|^s (\tilde{H}^{-1/s'} w^{p'/s'})^s dx\right)^{1/s}. \end{aligned}$$

From item 3 of the Lemma 3.1, $\tilde{H}w^{-p'} \in A_1^\rho$ and so, $\tilde{H}^{-1/s'} w^{p'/s'} \in A_{s,\infty}^\rho$. Thus, based on the hypothesis, the Lemma 3.2 and (4.1) we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |g|^p w^p dx\right)^{1/p} &\geq \left(\int_{\mathbb{R}^n} |g|^s \left(\tilde{H}^{-1/s'} w^{p'/s'}\right)^s dx\right)^{1/s} \\ &\geq C \|f\|_{\mathcal{L}_\delta(\tilde{H}^{-1/s'} w^{p'/s'})} \\ &\geq C \|f_\delta^{\sharp,\rho} \tilde{H}^{-1/s'} w^{p'/s'}\|_\infty \\ &\geq C \|f_\delta^{\sharp,\rho} \tilde{H}^{-1/s'} w^{p'/s'}\|_\infty \|\tilde{H}^{1/s'} w^{-p'/r}\|_r \\ &\geq C \|f_\delta^{\sharp,\rho} w\|_r. \end{aligned}$$

Moreover, considering that $w \in A_{p,q}^\rho$ and $r > p$, we obtain $w \in A_{r,q}^\rho$. Then, by Proposition 3.4 and the fact that $A_{r,q}^\rho \subset A_{r,q}^{\rho,\text{loc}}$, we have

$$\left(\int_{\mathbb{R}^n} |g|^p w^p dx\right)^{1/p} \geq C \|f_\delta^{\sharp,\rho} w\|_r \geq C \|f_0^{\sharp,\rho} w\|_q$$

because $\frac{1}{q} = \frac{1}{r} - \frac{\delta}{n}$.

Finally, since $w \in A_{p,q}^\rho$, it follows that $w^q \in A_{1+\frac{q}{p}}^\rho \subset A_{1+\frac{q}{p}}^{\rho,\text{loc}}$ and thus, by the Lemma 3.3, we get that

$$\|f_0^{\sharp,\rho} w\|_q \geq C \|fw\|_q,$$

which concludes the proof. □

Proof of Theorem 2.5. Let $w \in A_{p,\infty}^\rho$ and suppose as before that $\|gw\|_p < \infty$. Let us take r such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{s} = \frac{\delta-\delta'}{n}$.

By items 3. and 1. of Proposition 2.1, since $w \in A_{p,\infty}^\rho$, $w^{-p'} \in A_1^\rho$ and thus $w^{-p'} \in A_{1+p'/r}^\rho$.

In the same way as for the previous theorem, given g we obtain a function $\tilde{H} \in L^{r/s'}(w^{-p'})$ such that

$$\|\tilde{H}^{1/s'} w^{-p'/r}\|_r \leq C,$$

with $\tilde{H}^{-1/s'} w^{p'/s'} \in A_{s,\infty}^\rho$ and thereafter, from the hypothesis

$$\left(\int_{\mathbb{R}^n} |g|^p w^p dx \right)^{1/p} \geq C \|f_\delta^{\sharp,\rho} w\|_r.$$

Now, since $w \in A_{p,\infty}^\rho$ and $r \geq p$, by item 6. of Proposition 2.1 we have $w \in A_{r,\infty}^\rho$. Then, since $A_{r,\infty}^\rho \subset A_{r,\infty}^{\rho,\text{loc}}$, applying Proposition 3.4 and Lemma 3.2 we get that

$$\left(\int_{\mathbb{R}^n} |g|^p w^p dx \right)^{1/p} \geq C \|f_\delta^{\sharp,\rho} w\|_r \geq C \|f_\delta^{\sharp,\rho} w\|_\infty \geq \|f\|_{\mathcal{L}_\delta(w)}.$$

□

Proof of Theorem 2.15. Let f such that $\|f\|_{L^{q(\cdot)}(w)} < \infty$ and suppose as before that $0 < \|g\|_{L^{p(\cdot)}(w)} < \infty$.

Let us take

$$h_1 = \frac{f}{\|fw\|_{q(\cdot)}} + w^{-1} \left(\frac{g w}{\|gw\|_{p(\cdot)}} \right)^{\frac{p(\cdot)}{q(\cdot)}}.$$

Let us first see that $\|h_1 w\|_{q(\cdot)} \leq C$. Indeed, from the definition of the norm, it follows that

$$\begin{aligned} \varrho_{q(\cdot)}(h_1 w) &= \int_{\mathbb{R}^n} |h_1(x)w(x)|^{q(x)} dx = \int_{\mathbb{R}^n} \left| \frac{f(x)w(x)}{\|fw\|_{q(\cdot)}} + \left(\frac{g(x)w(x)}{\|gw\|_{p(\cdot)}} \right)^{\frac{p(x)}{q(x)}} \right|^{q(x)} dx \\ &\leq 2^{q^+} \int_{\mathbb{R}^n} \left(\frac{|f(x)w(x)|}{\|fw\|_{q(\cdot)}} \right)^{q(x)} dx + 2^{q^+} \int_{\mathbb{R}^n} \left(\frac{|g(x)w(x)|}{\|gw\|_{p(\cdot)}} \right)^{p(x)} dx \\ &\leq 2^{q^++1}. \end{aligned}$$

So, considering Proposition 2.6 it follows what has been stated.

We will now use the Corollary 3.13 and the operators H_1 and H_2 defined there for h_1 and the following h_2 . Let $r_0 = q_0/\sigma$. By Proposition 2.7, there exists $h_2 \in L^{(q(\cdot)/\sigma)'(\mathbb{R}^n)}$, $h_2 \geq 0$, $\|h_2\|_{(q(\cdot)/\sigma)'} \leq 1$, such that for any $\gamma > 0$,

$$\begin{aligned} \|fw\|_{q(\cdot)}^\sigma &= \|f^\sigma w^\sigma\|_{q(\cdot)/\sigma} \leq C \int_{\mathbb{R}^n} f^\sigma w^\sigma h_2 dx \leq C \int_{\mathbb{R}^n} f^\sigma H_1^{-\gamma} H_1^\gamma w^\sigma H_2 dx \\ &\leq C \left(\int_{\mathbb{R}^n} f^{q_0} H_1^{-\gamma r_0} H_2 w^\sigma dx \right)^{1/r_0} \left(\int_{\mathbb{R}^n} H_1^{\gamma r'_0} H_2 w^\sigma dx \right)^{1/r'_0} \\ &= CA^{1/r_0} B^{1/r'_0}, \end{aligned}$$

where we have consecutively considered Proposition 2.9, the pointwise inequality between h_2 and H_2 and the Hölder inequality with exponent r_0 and measure $w^\sigma H_2 dx$.

Let us now verify that the factor B is uniformly bounded. On the one hand, considering the variable Hölder inequality (Proposition 2.8) with exponent $q(\cdot)/\sigma$, we have that

$$B \leq C \|H_1^{\gamma r'_0} w^\sigma\|_{q(\cdot)/\sigma} \|H_2\|_{(q(\cdot)/\sigma)'}$$

Thus, since $h_1 \in L^{q(\cdot)}(w)$ and $h_2 \in L^{(q(\cdot)/\sigma)'(\mathbb{R}^n)}$, taking $\gamma = \frac{\sigma}{r'_0}$ and considering again the Corollary 3.13 and Proposition 2.9 we can see that

$$\|H_1^{\gamma r'_0} w^\sigma\|_{q(\cdot)/\sigma} = \|H_1^\sigma w^\sigma\|_{q(\cdot)/\sigma} = \|H_1 w\|_{q(\cdot)}^\sigma \leq 2^\sigma \|h_1 w\|_{q(\cdot)}^\sigma \leq C,$$

and

$$\|H_2\|_{(q(\cdot)/\sigma)'} \leq 2\|h_2\|_{(q(\cdot)/\sigma)'} \leq C.$$

Let us now observe that from the choice of γ , we have that

$$A = \int_{\mathbb{R}^n} f^{q_0} H_1^{-q_0/r'_0} H_2 w^\sigma dx.$$

In order to use the hypothesis (2.7), we must jointly prove that A is finite and that the weight $w_0 = (H_1^{-q_0/r'_0} H_2 w^\sigma)^{1/q_0}$ belongs to A_{p_0, q_0}^ρ .

Let's start by seeing that A is finite. Considering the definition of h_1 , the properties of H_1 and H_2 and the variable Hölder inequality it follows that

$$\begin{aligned} A &= \int_{\mathbb{R}^n} f^{q_0} H_1^{-q_0} H_1^{q_0} H_1^{-q_0/r'_0} H_2 w^\sigma dx \\ &\leq \int_{\mathbb{R}^n} f^{q_0} h_1^{-q_0} H_1^{q_0} H_1^{-q_0/r'_0} H_2 w^\sigma dx \\ &\leq \int_{\mathbb{R}^n} f^{q_0} \left(\frac{f}{\|fw\|_{q(\cdot)}} \right)^{-q_0} H_1^{q_0} H_1^{-q_0/r'_0} H_2 w^\sigma dx \\ &= \|fw\|_{q(\cdot)}^{q_0} \int_{\mathbb{R}^n} H_1^{q_0} H_1^{-q_0/r'_0} H_2 w^\sigma dx \\ &= \|fw\|_{q(\cdot)}^{q_0} \int_{\mathbb{R}^n} H_1^\sigma H_2 w^\sigma dx \\ &\leq C \|fw\|_{q(\cdot)}^{q_0} \|H_1^\sigma w^\sigma\|_{q(\cdot)/\sigma} \|H_2\|_{(q(\cdot)/\sigma)'} < \infty. \end{aligned}$$

Let us now see that $w_0 = (H_1^{-q_0/r'_0} H_2 w^\sigma)^{1/q_0} \in A_{p_0, q_0}^\rho$. For this purpose, considering the item 4. of Proposition 2.1, it is sufficient to prove that $w_0^{q_0} = H_1^{-(q_0-\sigma)} H_2 w^\sigma \in A_{s_0}^\rho$, where

$$s_0 = 1 + \frac{q_0}{p'_0} = \frac{q_0}{\sigma}.$$

In view again of Proposition 2.1, but in this case item 2., given that both $H_2 w^\sigma$ and $H_1^{\frac{q_0-\sigma}{s_0-1}} = H_1^\sigma$ belong to A_1^ρ , it follows immediately that

$$w_0^{q_0} = \left(H_1^{\frac{q_0-\sigma}{s_0-1}} \right)^{1-s_0} H_2 w^\sigma \in A_{s_0}^\rho.$$

We are now in a position to apply (2.7). Considering then the hypothesis, the definition of h_1 and Hölder's inequality with respect to a certain exponent $r(\cdot)$, we

have that

$$\begin{aligned}
 A^{1/r_0} &= A^{\sigma/q_0} = \left(\int_{\mathbb{R}^n} f^{q_0} H_1^{-q_0/r'_0} H_2 w^\sigma dx \right)^{\sigma/q_0} \\
 &\leq C \left(\int_{\mathbb{R}^n} g^{p_0} (H_1^{-q_0/r'_0} H_2 w^\sigma)^{p_0/q_0} dx \right)^{\sigma/p_0} \\
 &\leq C \|gw\|_{p(\cdot)}^\sigma \left(\int_{\mathbb{R}^n} \left(h_1^{\frac{q(\cdot)}{p(\cdot)}} w^{\frac{q(\cdot)}{p(\cdot)}-1} \right)^{p_0} H_1^{-p_0/r'_0} H_2^{p_0/q_0} w^{\sigma p_0/q_0} dx \right)^{\sigma/p_0} \\
 &\leq C \|gw\|_{p(\cdot)}^\sigma \left(\int_{\mathbb{R}^n} H_1^{p_0 \left(\frac{q(\cdot)}{p(\cdot)} - \frac{1}{r'_0} \right)} w^{p_0 \left(\frac{\sigma}{q_0} + \frac{q(\cdot)}{p(\cdot)} - 1 \right)} H_2^{p_0/q_0} dx \right)^{\sigma/p_0} \\
 &\leq C \|gw\|_{p(\cdot)}^\sigma \left\| (H_1 w)^{p_0 \left(\frac{q(\cdot)}{p(\cdot)} - \frac{1}{r'_0} \right)} \right\|_{r'(\cdot)}^{\sigma/p_0} \left\| H_2^{p_0/q_0} \right\|_{r(\cdot)}^{\sigma/p_0} \\
 &= C \|gw\|_{p(\cdot)}^\sigma E^{\sigma/p_0} F^{\sigma/p_0},
 \end{aligned}$$

where we have used the fact that $\frac{\sigma}{q_0} - 1 = -\frac{1}{r'_0}$.

Let us take $r(\cdot) = \frac{q_0}{p_0} \left(\frac{q(\cdot)}{\sigma} \right)'$ and see that both E and F are bounded. On the one hand, considering the Proposition 2.9, we get that

$$F = \left\| H_2^{p_0/q_0} \right\|_{r(\cdot)} = \left\| H_2^{p_0/q_0} \right\|_{\frac{q_0}{p_0} \left(\frac{q(\cdot)}{\sigma} \right)'} = \|H_2\|_{(q(\cdot)/\sigma)'}^{p_0/q_0} \leq 2^{p_0/q_0} \|h_2\|_{(q(\cdot)/\sigma)'}^{p_0/q_0} \leq 2^{p_0/q_0}.$$

On the other hand, since

$$p_0 \left(\frac{q(\cdot)}{p(\cdot)} - \frac{1}{r'_0} \right) r'(\cdot) = q(\cdot),$$

we have that

$$\begin{aligned}
 \varrho_{r'(\cdot)} \left((H_1 w)^{p_0 \left(\frac{q(\cdot)}{p(\cdot)} - \frac{1}{r'_0} \right)} \right) &= \int_{\mathbb{R}^n} (H_1 w)^{p_0 \left(\frac{q(\cdot)}{p(\cdot)} - \frac{1}{r'_0} \right) r'(\cdot)} dx \\
 &= \int_{\mathbb{R}^n} (H_1 w)^{q(\cdot)} dx = \varrho_{q(\cdot)}(H_1 w).
 \end{aligned}$$

Given that $\|H_1 w\|_{q(\cdot)} \leq 2 \|h_1 w\|_{q(\cdot)}$, applying Proposition 2.6 it follows that $\varrho_{q(\cdot)}(H_1 w)$ is uniformly bounded, and therefore, applying Proposition 2.6 again, the uniform bounding of E follows, which concludes the proof of the theorem. \square

Proof of Theorem 2.17. Suppose as before that $0 < \|g\|_{L^{p(\cdot)}(w)} < \infty$.

$$\text{Let } \frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{1}{s} = \frac{\delta - \delta(\cdot)}{n}, \bar{q} = \bar{q}(\cdot) = \frac{q(\cdot)}{\eta}, \bar{p} = \bar{p}(\cdot) = \frac{p(\cdot)}{\eta} \text{ and } \bar{s} = \frac{s}{\eta}.$$

Let us consider the functions h and \tilde{h} defined by

$$h = \left(\frac{|g|^\eta w^{\eta \bar{p}'}}{\|gw\|_{p(\cdot)}^\eta} \right)^{\bar{p} - \bar{s}} = \left(\frac{|g| w^{\bar{p}'}}{\|gw\|_{p(\cdot)}} \right)^{p(\cdot) - s} \quad \text{and} \quad \tilde{h} = h^{-\bar{s}'/\bar{s}}.$$

From its definition, it follows that, $h \in L^{(p(\cdot)/s)'}(w^{-\eta\bar{p}'/(p(\cdot)/s)'})$ and

$$\|h w^{-\eta\bar{p}'/(p(\cdot)/s)'}\|_{(p(\cdot)/s)'} \leq 1.$$

Indeed, since $p(\cdot) = \left(\frac{p(\cdot)}{s}\right)'(p(\cdot) - s) = \bar{p}'(p(\cdot) - \eta)$ we have that

$$\begin{aligned} \int_{\mathbb{R}^n} h(x)^{(p(x)/s)'} w(x)^{-\eta\bar{p}'} dx &= \int_{\mathbb{R}^n} \left(\frac{|g(x)|w(x)^{\bar{p}'}}{\|gw\|_{p(\cdot)}} \right)^{p(x)} w(x)^{-\eta\bar{p}'} dx \\ &= \int_{\mathbb{R}^n} \frac{|g(x)w(x)|^{p(x)}}{\|gw\|_{p(\cdot)}^{p(x)}} dx \leq 1. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^n} \tilde{h}(x)^{\bar{q}/\bar{s}'} w(x)^{-\eta\bar{p}'} dx &= \int_{\mathbb{R}^n} h(x)^{-q(x)/s} w(x)^{-\eta\bar{p}'} dx \\ &= \int_{\mathbb{R}^n} h(x)^{(p(x)/s)'} w(x)^{-\eta\bar{p}'} dx \leq 1, \end{aligned}$$

that is $\tilde{h} \in L^{\bar{q}/\bar{s}'}(w^{-\eta\bar{p}'\bar{s}'/\bar{q}})$ and $\|\tilde{h}w^{-\eta\bar{p}'\bar{s}'/\bar{q}}\|_{\bar{q}/\bar{s}'} \leq 1$.

Moreover,

$$\begin{aligned} \int_{\mathbb{R}^n} |g(x)^\eta w(x)^{\eta\bar{p}'}|^{\bar{s}} |\tilde{h}(x)^{-\bar{s}/\bar{s}'} w(x)^{-\eta\bar{p}'}| dx &= \int_{\mathbb{R}^n} |g(x)^\eta w(x)^{\eta\bar{p}'}|^{\bar{s}} h(x) w(x)^{-\eta\bar{p}'} dx \\ &= \int_{\mathbb{R}^n} \frac{(|g(x)|w(x)^{\bar{p}'})^{p(x)}}{\|gw\|_{p(\cdot)}^{p(x)-s}} w(x)^{-\eta\bar{p}'} dx \\ &= \int_{\mathbb{R}^n} \frac{(|g(x)|w(x))^{p(x)}}{\|gw\|_{p(\cdot)}^{p(x)-s}} dx \\ &\leq \|gw\|_{p(\cdot)}^s. \end{aligned} \tag{4.2}$$

Note that, since $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{1}{s}$, $w^\eta \in A_{p(\cdot)/\eta, \infty}^\rho$ and $\tilde{h} \in L^{\bar{q}/\bar{s}'}(w^{-\eta\bar{p}'\bar{s}'/\bar{q}})$, by Corollary 3.14 there exists $\tilde{H} \in L^{\bar{q}/\bar{s}'}(w^{-\eta\bar{p}'\bar{s}'/\bar{q}})$ such that

- (1) $\tilde{h}(x) \leq \tilde{H}(x)$ a.e. $x \in \mathbb{R}^n$.
- (2) $\|\tilde{H}w^{-\eta\bar{p}'\bar{s}'/\bar{q}}\|_{\bar{q}/\bar{s}'} \leq 2\|\tilde{h}w^{-\eta\bar{p}'\bar{s}'/\bar{q}}\|_{\bar{q}/\bar{s}'}$.
- (3) $\tilde{H}w^{-\eta\bar{p}'} \in A_1^\rho$.

Note that from item 2 and the fact that $\|\tilde{h}w^{-\eta\bar{p}'\bar{s}'/\bar{q}}\|_{\bar{q}/\bar{s}'} \leq 1$, it follows that $\|\tilde{H}w^{-\eta\bar{p}'\bar{s}'/\bar{q}}\|_{\bar{q}/\bar{s}'} \lesssim C$, which by Proposition 2.9, is equivalent to

$$\|\tilde{H}^{1/(\eta\bar{s}')}w^{-\bar{p}'/\bar{q}}\|_{q(\cdot)} \lesssim C. \tag{4.3}$$

From (4.2) and item 1 we obtain

$$\begin{aligned} \|gw\|_{p(\cdot)} &\geq \left(\int_{\mathbb{R}^n} |g(x)^\eta w(x)^{\eta\bar{p}'/\bar{s}} \tilde{H}(x)^{-\bar{s}/\bar{s}'} w(x)^{-\eta\bar{p}'} dx \right)^{1/s} \\ &\geq \left(\int_{\mathbb{R}^n} |g(x)|^s \left(\tilde{H}(x)^{-1/(\eta\bar{s}')} w(x)^{\bar{p}'/\bar{s}'} \right)^s dx \right)^{1/s} \end{aligned}$$

On the other hand, from item 3 we get that the weight $(\tilde{H}^{-1/\bar{s}'} w^{\eta\bar{p}'/\bar{s}'})^{-\bar{s}'} = \tilde{H} w^{-\eta\bar{p}'} \in A_1^\rho$, which is equivalent, by Proposition 2.1 (3), to $\tilde{H}^{-1/\bar{s}'} w^{\eta\bar{p}'/\bar{s}'} \in A_{\bar{s},\infty}^\rho$, that is, $(\tilde{H}^{-1/(\eta\bar{s}')} w^{\bar{p}'/\bar{s}'})^\eta \in A_{s/\eta,\infty}^\rho$. Thus, by the hypothesis, Lemma 3.2 and (4.3) we obtain

$$\begin{aligned} \|gw\|_{p(\cdot)} &\geq C \|f\|_{\mathcal{L}_\delta(\tilde{H}^{-1/(\eta\bar{s}')} w^{\bar{p}'/\bar{s}'})} \\ &\geq C \|f_\delta^{\sharp,\rho} \tilde{H}^{-1/(\eta\bar{s}')} w^{\bar{p}'/\bar{s}'}\|_\infty \\ &\geq C \|f_\delta^{\sharp,\rho} \tilde{H}^{-1/(\eta\bar{s}')} w^{\bar{p}'/\bar{s}'}\|_\infty \|\tilde{H}^{1/(\eta\bar{s}')} w^{-\bar{p}'/\bar{q}}\|_{q(\cdot)} \\ &\geq C \|f_\delta^{\sharp,\rho} w^{\bar{p}'(1/\bar{s}'-1/\bar{q})}\|_{q(\cdot)} \\ &= C \|f_\delta^{\sharp,\rho} w\|_{q(\cdot)} \end{aligned}$$

From the hypothesis on w and Proposition 3.8 we conclude that $w \in A_{q(\cdot),\infty}^\rho$ with $q(\cdot) = \frac{n}{\delta-\delta(\cdot)}$. Thus, from Proposition 3.16 and Lemma 3.15 we get that

$$\begin{aligned} \|gw\|_{p(\cdot)} &\geq C \|f_\delta^{\sharp,\rho} w\|_{q(\cdot)} \\ &= C \|f_\delta^{\sharp,\rho} w\|_{\frac{n}{\delta-\delta(\cdot)}} \\ &\geq C \|f_{\delta(\cdot),p(\cdot)}^{\sharp,\rho} w\|_\infty \\ &\geq C \|f\|_{\mathcal{L}_{\delta(\cdot)}(w)}. \end{aligned}$$

□

5. APPLICATIONS

We will conclude this paper with some results where we essentially prove norm inequalities on the weighted variable Lebesgue spaces for some fractional type operators in the Schrödinger context.

First we will see how to prove that an operator T is bounded in $L^{p(\cdot)}(w)$ using Theorem 2.15. These same ideas can be used to apply the other theorems.

The key point in applying Theorem 4 is to consider an appropriate \mathcal{F} family. This usually requires a density argument, since we need pairs of functions (f, g) such that f lies both the appropriate weighted space to apply the hypothesis and in the weighted variable Lebesgue space in which we want to obtain the thesis.

The dense subsets of $L^p(w)$ are well known, for example, smooth functions and bounded functions of compact support.

These sets are also dense in $L^{p(\cdot)}(\mathbb{R}^n)$ (see for example [13]) and in $L^{p(\cdot)}(w)$ (see [12]).

More specifically, in [12] it is proved that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $p^+ < \infty$ and $w \in L^{p(\cdot)}_{loc}(\mathbb{R}^n)$, then $L^\infty_c(\mathbb{R}^n)$, the set of bounded functions of compact support, and C^∞_c , the smooth functions of compact support, are dense in $L^{p(\cdot)}(w)$.

Suppose now that for all $w_0 \in A^p_{p_0, q_0}$ it holds that

$$\|Tf w_0\|_{q_0} \leq C \|f w_0\|_{p_0}. \tag{5.1}$$

We want to show that given a $w \in A^p_{p(\cdot), q(\cdot)}$, T maps $L^{p(\cdot)}(w)$ into $L^{q(\cdot)}(w)$. Since $w \in L^{p(\cdot)}_{loc}(\mathbb{R}^n)$ and $w \in L^{q(\cdot)}_{loc}(\mathbb{R}^n)$, by a standard density argument (see [13, Theorem 5.39]) it is sufficient to show that

$$\|Tf w\|_{q(\cdot)} \leq C \|f w\|_{p(\cdot)},$$

for all $f \in L^\infty_c$. Although intuitively, it can be thought to define \mathcal{F} as

$$\mathcal{F} = \{(|Tf|, |f|) : f \in L^\infty_c\},$$

it is not known a priori that Tf is in $L^{q(\cdot)}(w)$. To overcome this, we again proceed by approximation and define $(Tf)_n = \min\{|Tf|, n\} \chi_{B(0, n)}$. Given that $w \in L^{p(\cdot)}_{loc}(\mathbb{R}^n)$, it follows that $(Tf)_n \in L^{q(\cdot)}(w)$. On the other hand, it is clear that (5.1) is satisfied with $|Tf|$ replaced by $(Tf)_n$. Therefore, if we define

$$\mathcal{F} = \{((Tf)_n, |f|) : f \in L^\infty_c, n \geq 1\},$$

we can apply Theorem 2.15 and Fatou's Lemma in this context (see [13], Theorem 2.61) and conclude that for any $f \in L^\infty_c$

$$\|Tf w\|_{q(\cdot)} \leq \liminf_{n \rightarrow \infty} \|(Tf)_n w\|_{q(\cdot)} \leq C \|f w\|_{p(\cdot)}.$$

5.1. Fractional integral. Given $0 < \alpha < n$, the fractional integral of order α associated with \mathcal{L} is defined as

$$\mathcal{I}_\alpha f(x) = \mathcal{L}^{-\alpha/2} f(x) = \int_{\mathbb{R}^n} K_\alpha(x, y) f(y) dy,$$

where

$$K_\alpha(x, y) = \int_0^\infty k_t(x, y) t^{\alpha/2} \frac{dt}{t}.$$

Here k_t denotes the kernel of $\{e^{-t\mathcal{L}}\}_{t>0}$, the heat semigroup associated to \mathcal{L} . It is known (see [22] and [18]) that if $V \in RH_\nu$ with $\nu > n/2$ and $\lambda_0 = \min\{1, 2 - \frac{n}{\nu}\}$, given $N > 0$ and $0 < \lambda < \lambda_0$, there exists a constant C such that

$$k_t(x, y) \leq C t^{-n/2} e^{-\frac{|x-y|^2}{Ct}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}$$

for any x and y in \mathbb{R}^n . Also

$$|k_t(x, y) - k_t(x_0, y)| \leq C \left(\frac{|x - x_0|}{\sqrt{t}}\right)^\lambda t^{-n/2} e^{-\frac{|x-y|^2}{Ct}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N},$$

provided that $|x - x_0| < \sqrt{t}$.

Taking into account the above estimations, the following bounding result was proved in [2, Proposition 8].

Proposition 5.1. *If $V \in RH_\nu$ with $\nu > n/2$, then given $N > 0$ and $0 < \lambda < \lambda_0$, there exists a constant C such that,*

$$|K_\alpha(x, y)| \leq \frac{C}{|x - y|^{n-\alpha}} \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N}, \quad x, y \in \mathbb{R}^n, \quad (5.2)$$

and

$$|K_\alpha(x, y) - K_\alpha(x_0, y)| \leq C \frac{|x - x_0|^\lambda}{|x - y|^{n-\alpha+\lambda}} \left(1 + \frac{|x - y|}{\rho(x_0)}\right)^{-N}, \quad (5.3)$$

whenever $|x - x_0| < \frac{1}{2}|x_0 - y|$.

Theorem 5.2. *Let $V \in RH_\nu$ with $\nu > \frac{n}{2}$, $0 < \alpha < n$, $0 \leq \delta < \min\{\lambda_0, \alpha\}$ and s such that $\frac{1}{s} = \frac{\alpha}{n} - \frac{\delta}{n}$. Then, there exists a constant C such that the inequality*

$$\|\mathcal{I}_\alpha f\|_{\mathcal{L}_\delta(w)} \leq C \|f\|_{L^s(w^s)}$$

is satisfied for all $w \in A_{s,\infty}^\rho$.

Proof. Let $x_0 \in \mathbb{R}^n$, and consider $\tilde{B} = B(x_0, R)$ with $R \leq \rho(x_0)$. From (5.2), the Hölder inequality and since $w^{-s'} \in A_1^\rho$, it follows that

$$\begin{aligned} \frac{1}{|\tilde{B}|} \int_{\tilde{B}} |\mathcal{I}_\alpha(f\chi_{2\tilde{B}})(x)| dx &\lesssim \frac{1}{|\tilde{B}|} \int_{\tilde{B}} \int_{2\tilde{B}} |K_\alpha(x, y)| |f(y)| dy dx \\ &\lesssim \frac{1}{|\tilde{B}|} \int_{\tilde{B}} \int_{2\tilde{B}} \frac{|f(y)|}{|x - y|^{n-\alpha}} \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N} dy dx \\ &\lesssim \frac{1}{|\tilde{B}|} \int_{2\tilde{B}} |f(y)| \left(\int_{B(y, 3R)} \frac{1}{|x - y|^{n-\alpha}} dx \right) dy \\ &\lesssim |\tilde{B}|^{\alpha/n-1} \int_{2\tilde{B}} |f(y)| dy \\ &\lesssim |\tilde{B}|^{\alpha/n-1} \|fw\|_s \left(\int_{2\tilde{B}} w(y)^{-s'} dy \right)^{1/s'} \\ &\lesssim |\tilde{B}|^{\alpha/n-1} |\tilde{B}|^{1/s'} \frac{\|fw\|_s}{\|w\chi_{2\tilde{B}}\|_\infty} \left(1 + \frac{2R}{\rho(x_0)}\right)^{\theta/s'} \\ &\leq C |\tilde{B}|^{\delta/n} \frac{\|fw\|_s}{\|w\chi_{\tilde{B}}\|_\infty}. \end{aligned} \quad (5.4)$$

We will check first condition (2.4). We write $f = \tilde{f}_1 + \tilde{f}_2$, with $\tilde{f}_1 = f\chi_{2B(x_0, \rho(x_0))}$. The observation above with $R = \rho(x_0)$ gives us the estimate we need for \tilde{f}_1 . For $x \in B(x_0, \rho(x_0))$ and $y \in (2B(x_0, \rho(x)))^c$, we have $\rho(x) \simeq \rho(x_0)$ and $|x_0 - y| \leq 2|x - y|$. Then, calling $B'_k = B(x_0, 2^k \rho(x_0))$, by using estimate (5.2) with $N > \delta + \frac{\theta}{s'}$ and Hölder's inequality, we get, for $x \in B(x_0, \rho(x_0))$,

$$\begin{aligned}
 |\mathcal{I}_\alpha(\tilde{f}_2)(x)| &\lesssim \int_{(2B(x_0, \rho(x_0)))^c} \frac{|f(y)|}{|x-y|^{n-\alpha}} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} dy \\
 &\lesssim \sum_{k \geq 1} \int_{B'_{k+1} \setminus B'_k} \frac{|f(y)|}{|x_0-y|^{n-\alpha}} \left(1 + c \frac{|x_0-y|}{\rho(x_0)}\right)^{-N} dy \\
 &\lesssim \frac{1}{\rho(x_0)^{n-\alpha}} \sum_{k \geq 1} \frac{1}{2^{k(n-\alpha)}} \left(1 + \frac{2^k \rho(x_0)}{\rho(x_0)}\right)^{-N} \int_{B'_{k+1}} |f(y)| dy \\
 &\lesssim \frac{\|fw\|_s}{\rho(x_0)^{n-\alpha}} \sum_{k \geq 1} \frac{2^{-kN}}{2^{k(n-\alpha)}} \left(\int_{B'_{k+1}} w(y)^{-s'} dy\right)^{1/s'} \\
 &\lesssim \frac{\|fw\|_s}{\rho(x_0)^{n-\alpha}} \sum_{k \geq 1} \frac{2^{-k(N-\frac{\theta}{s'})}}{2^{k(n-\alpha)}} |B'_{k+1}|^{1/s'} \left(\inf_{B'_{k+1}} w^{-s'}\right)^{1/s'} \\
 &\lesssim \frac{\|fw\|_s}{\rho(x_0)^{n-\alpha-\frac{n}{s'}}} \sum_{k \geq 1} \frac{2^{-k(N-\frac{\theta+n}{s'})}}{2^{k(n-\alpha)}} \frac{1}{\|w\chi_{B'_{k+1}}\|_\infty} \\
 &\lesssim \frac{\|fw\|_s}{\|w\chi_{B(x_0, \rho(x_0))}\|_\infty} \rho(x_0)^{\alpha-\frac{n}{s'}} \left(\sum_{k \geq 1} 2^{-k[N-(\alpha-\frac{n}{s'})-\frac{\theta}{s'}]}\right) \\
 &\lesssim \frac{\|fw\|_s}{\|w\chi_{B(x_0, \rho(x_0))}\|_\infty} |B(x_0, \rho(x_0))|^{\delta/n}.
 \end{aligned}$$

Now, we must check condition (2.3). It will be enough to show that this inequality is satisfied for every ball $B \in \mathcal{B}_\rho$ for some constant c_B . Let $B = B(x_0, r)$ with $r \leq \rho(x_0)$ and we write $f = f_1 + f_2$, with $f_1 = f\chi_{2B}$. As before, the estimate for f_1 follows from (5.4) for $R = r$.

Finally we will estimate $|\mathcal{I}_\alpha(f_2)(x) - c_B|$, uniformly for $x \in B$, with $c_B = \mathcal{I}_\alpha(f_2)(x_0)$. For $x \in B$ and $y \in (2B)^c$, we have $\rho(x) \simeq \rho(x_0)$ and $|x_0 - y| \leq 2|x - y|$. Then, calling $B_k = B(x_0, 2^k r)$, by using estimate (5.3) with $N = \frac{\theta}{s'}$ and $\delta < \lambda < \lambda_0$, we have

$$\begin{aligned}
 |\mathcal{I}_\alpha(f_2)(x) - \mathcal{I}_\alpha(f_2)(x_0)| &\lesssim \int_{(2B)^c} |f(y)| \frac{|x-x_0|^\lambda}{|x-y|^{n-\alpha+\lambda}} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} dy \\
 &\lesssim r^\lambda \sum_{k \geq 1} \int_{B_{k+1} \setminus B_k} \frac{|f(y)|}{|x_0-y|^{n-\alpha+\lambda}} \left(1 + c \frac{|x_0-y|}{\rho(x_0)}\right)^{-N} dy \\
 &\lesssim \frac{1}{r^{n-\alpha}} \sum_{k \geq 1} \frac{1}{2^{k(n-\alpha+\lambda)}} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{-N} \int_{B_{k+1}} |f(y)| dy \\
 &\lesssim \frac{\|fw\|_s}{r^{n-\alpha}} \sum_{k \geq 1} \frac{1}{2^{k(n-\alpha+\lambda)}} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{-N} \left(\int_{B_{k+1}} w(y)^{-s'} dy\right)^{1/s'}
 \end{aligned}$$

$$\begin{aligned}
 &\lesssim \frac{\|fw\|_s}{r^{n-\alpha}} \sum_{k \geq 1} \frac{|B_{k+1}|^{1/s'}}{2^{k(n-\alpha+\lambda)}} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{-N + \frac{\theta}{s'}} \left(\inf_{B_{k+1}} w^{-s'}\right)^{1/s'} \\
 &\lesssim \frac{\|fw\|_s}{r^{n-\alpha-\frac{n}{s'}}} \sum_{k \geq 1} \frac{2^{k\frac{n}{s'}}}{2^{k(n-\alpha+\lambda)}} \frac{1}{\|w\chi_{B_{k+1}}\|_\infty} \\
 &\lesssim \frac{\|fw\|_s}{\|w\chi_B\|_\infty} r^\delta \left(\sum_{k \geq 1} 2^{-k(\lambda-\delta)}\right) \\
 &\lesssim \frac{\|fw\|_s}{\|w\chi_B\|_\infty} |B|^{\delta/n}. \quad \square
 \end{aligned}$$

Remark 5.3. The above result was proved for the case $\delta = 0$ in [1].

Thus, we can apply the Theorem 2.16 to get the following.

Theorem 5.4. Let $V \in RH_\nu$ con $\nu > \frac{n}{2}$, $0 < \alpha < n$ and $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that

$$\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n}.$$

Then, if $w \in A_{p(\cdot), q(\cdot)}^\rho$ and $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $q^- > \frac{n}{n-\alpha}$, it follows that

$$\|\mathcal{I}_\alpha f\|_{L^{q(\cdot)}(w)} \leq C \|f\|_{L^{p(\cdot)}(w)}.$$

Remark 5.5. Since $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n}$ we have that $\frac{1}{p^+} - \frac{1}{q^+} = \frac{\alpha}{n}$ from which it follows that $p^+ < \frac{n}{\alpha}$. The restriction $p^+ < \frac{n}{\alpha}$ is natural for the fractional integral operator, since in the constant exponent case \mathcal{I}_α does not map $L^{n/\alpha}$ to L^∞ .

On the other hand, also as a consequence of Theorem 5.2, now in combination with Theorem 2.17, the following result follows.

Theorem 5.6. Let $V \in RH_\nu$ with $\nu > \frac{n}{2}$, $0 \leq \delta < \min\{\alpha, \lambda_0\}$. Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $p^- > 1$, and $0 \leq \tilde{\delta}(\cdot) < \delta$ such that $\frac{1}{p(\cdot)} = \frac{\alpha}{n} - \frac{\tilde{\delta}(\cdot)}{n}$. Then, for every $w \in A_{p(\cdot), \infty}^\rho$ it follows that

$$\|\mathcal{I}_\alpha f\|_{\mathcal{L}_{\tilde{\delta}(\cdot)}(w)} \leq C \|f\|_{L^{p(\cdot)}(w)}.$$

5.2. Commutators of fractional integrals. We now approach the case of the commutator of the operator \mathcal{I}_α . For a function b we will consider the commutator of \mathcal{I}_α defined as,

$$[\mathcal{I}_\alpha, b]f(x) = \mathcal{I}_\alpha(bf)(x) - b(x)\mathcal{I}_\alpha f(x), \quad x \in \mathbb{R}^n.$$

It is known that for the classical case (i.e., $V \equiv 0$), the commutator of the \mathcal{I}_α operator is of strong type (p, q) for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ provided that b belongs to BMO , the bounded mean oscillation space.

For the case we are dealing with here, the operator $[\mathcal{I}_\alpha, b]$, it satisfies analogous bounding properties, but the class where b belongs is larger than the usual BMO .

Let us consider the space BMO_ρ , defined in [4], as the set of locally integrable functions b such that for some $\theta > 0$,

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy \leq C \left(1 + \frac{r}{\rho(x)}\right)^\theta,$$

for all $x \in \mathbb{R}^n$ and $r > 0$. The following result was proved in [27, Theorem 4.4].

Theorem 5.7. *Let $V \in RH_\nu$ with $\nu > \frac{n}{2}$ and $b \in BMO_\rho$. Then, if $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, there exists a constant C such that the inequality*

$$\|[\mathcal{I}_\alpha, b]f\|_{L^q(w^q)} \leq C \|f\|_{L^p(w^p)}$$

is satisfied for all $w \in A_{p,q}^\rho$.

Hence, in view of Theorem 2.15, we obtain the following bounding result.

Theorem 5.8. *Let $V \in RH_\nu$ with $\nu > \frac{n}{2}$, $b \in BMO_\rho$, $0 < \alpha < n$ and $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that*

$$\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n}.$$

Then, if $w \in A_{p(\cdot),q(\cdot)}^\rho$ and $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $q^- > \frac{n}{n-\alpha}$, it follows that

$$\|[\mathcal{I}_\alpha, b]f\|_{L^{q(\cdot)}(w)} \leq C \|f\|_{L^{p(\cdot)}(w)}.$$

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