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$L^p(\mathbb{R}^n)$-DIMENSION FREE ESTIMATES OF THE RIESZ TRANSFORMS

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ABSTRACT. In this note we describe some known results about dimension free boundedness in $L^p(\mathbb{R}^n)$ of the Riesz transforms, for p in the range $1 < p < \infty$.

Coincidí con Pola en 1983, en uno de los primeros congresos de Análisis Armónico celebrados en El Escorial (España). Pero no fue hasta finales de 1988 cuando comenzamos a colaborar. Esta colaboración, que duró hasta su fallecimiento, tuvo dos facetas. En primer lugar, realizar una investigación de alto nivel internacional en Análisis Matemático. En segundo lugar, fomentar las relaciones profesionales de matemáticos argentinos, en especial jóvenes, con investigadores de otros países. Coincidiendo con el profesor Santaló (ver minuto 17:26 y siguientes de [25]), ésta era una de sus mayores inquietudes. De hecho, era habitual recibir mensajes suyos, incluido uno enviado unos pocos días antes de su fallecimiento, relativos a posibles estancias de argentinos en Europa y de europeos en Argentina. Del éxito de esta segunda faceta dan fe las decenas de artículos, publicados en revistas internacionales, firmados conjuntamente por investigadores argentinos y europeos que se escribieron siguiendo su estela.

La profesora Harboure concebía la investigación como un trabajo exigente con altas metas, pero al mismo tiempo algo con lo que debían disfrutar tanto ella como los muchos colaboradores que tenía. Sus reuniones de trabajo siempre comenzaban con una charla amigable y relajada sobre el tiempo, las noticias, nuestras familias… A continuación de manera muy humilde presentaba las ideas que había tenido desde la reunión anterior. Si alguien sugería una línea alternativa, ella la exploraba, incluso aunque tuviese el convencimiento de que la suya era más acertada. La influencia de la llamada generación del 61–62 en la UBA (ver minuto 34:19 de [25]), fue sin duda fundamental en su forma de entender la dedicación a la ciencia.

Creo que las palabras del profesor L. A. Santaló escritas en la necrológica del profesor M. Balanzat, [20], se ajustan perfectamente al vacío que nos ha producido la muerte de la profesora Eleonor Harboure:

> *Debemos expresar el profundo dolor de toda la colectividad matemática por la desaparición de la para nosotros inolvidable, querida y admirada amiga y compañera.*

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Introduction

In this note we describe some known results about dimension free boundedness in $L^p(\mathbb{R}^n)$ of the Riesz transforms, for p in the range $1 < p < \infty$. Eleonor Harboure was very interested in this kind of results and in fact she proved some of them as we shall show.

In 1928 Marcel Riesz proved the following theorem, see [19].

Theorem. *Let f be a function in $L^p(-\infty,\infty), 1 < p < \infty$. The principal value of the integral*

$$\overline{f}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy \tag{0.1}$$

exists almost everywhere and moreover

$$\int_{-\infty}^{\infty} |\overline{f}(x)|^p \, dx \le M_p^p \int_{-\infty}^{\infty} |f(x)|^p \, dx;$$

the constant M_p depends on p.

The motivation of M. Riesz was the relation of the L^p-norms of the real and imaginary parts of an holomorphic function. The theorem has a byproduct the boundedness in L^p of the partial sum operators of the Fourier series. An alternative proof was given by A. P. Calderón in the 1940s. Both proofs can be found in the last editions of the book [26]. For a nice story about Calderón's proof see the foreword by R. Fefferman in Zygmund's book [26]. Nowadays, the principal value integral (0.1) is called "Hilbert transform". The one-dimensional result was extended, in 1968, to several variables by A. P. Calderón and A. Zygmund in the celebrated paper [7]. They proved the following theorem.

Theorem. *Let f be a function in $L^p(\mathbb{R}^n), 1 < p < \infty$. Let $K(x) = \dfrac{\Omega(x)}{|x|^n}$, $x \in \mathbb{R}^n$, be a function such that*

- *the mean value of Ω over the unit sphere is zero,*
- *the function $\Omega(x_1', x_2', \dots, x_n')$ satisfies a certain smoothness condition.*

Then the principal value of the integral

$$\mathcal{K}f(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\, dy$$

exists almost everywhere, converges in the L^p-norm and moreover

$$\int_{\mathbb{R}^n} |\mathcal{K}f(x)|^p \, dx \le M_p^p \int_{\mathbb{R}^n} |f(x)|^p \, dx; \tag{0.2}$$

the constant M_p depends on p, Ω and the dimension n.

It could be said that the main examples of the operators covered by the work of A. P. Calderón and A. Zygmund are the so-called Riesz transforms in $\mathbb{R}^n$,

$$\mathcal{R}_i f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{x_i - y_i}{|x-y|^{n+1}} f(y)\, dy, \quad i = 1, \dots, n.$$

The Riesz transforms can also be defined in $L^2(\mathbb{R}^n)$ by using the Fourier transform. In fact it can be seen that $\widehat{\mathcal{R}_i f}(\xi_i) = i(\xi_i/|\xi|)\hat{f}(\xi), \xi \in \mathbb{R}^n$. By functional calculus, the last expression can be written as $\mathcal{R}_i f = \partial_i(-\Delta)^{-1/2} f$.

E. Stein in 1983, see [21], proved that the Riesz transforms are bounded in $L^p(\mathbb{R}^n), 1 < p < \infty$, with the constant M_p^p, in (0.2), independent of the dimension n. He used Littlewood–Paley g-functions. By transference methods J. Duoandikoetxea and J. L. Rubio de Francia, see [11], gave an alternative proof of Stein's result. The result is also known for the Riesz transforms associated to the differential operator in $\mathbb{R}^n$ given by $\Delta - 2x \cdot \nabla$. In this case the natural measure is the gaussian measure, and the eigenvalues of the operator turn out to be the Hermite polynomials. Proofs of these results were given by G. Pisier, see [18] and C. Gutiérrez who used an extension of the Littlewood–Paley g-functions, see [15]. In this case, since the measure is finite, R. Gundy and P. A. Meyer obtained dimension free results using probabilistic methods, see [14] and [17]. Analogous results were proved in [10] for the case of the Heisenberg group.

In the present note we shall briefly discuss two different results about the dimension free $L^p(\mathbb{R}^n)$-estimates of the Riesz transforms:

(1) Case of the Riesz transforms associated to the Hermite operator, see [16].
(2) Dimension free estimates for the maximal operator of the truncations of the Riesz transforms, see [12].

1. Hermite operator

In order to include into a general framework the operators considered above, we need to introduce the concept of diffusion semigroup.

Definition 1.1. *Let $(\mathcal{M}, d\mu)$ be a measure space and $\{T_t\}_{t>0}$ a family of operators bounded in L^2 such that*

- $T_{t_1+t_2} = T_{t_1}T_{t_2}$. $T_0 = Id$. $\lim_{t\to 0} T_t f = f$ *in* L^2.
- $\|T_t f\|_p \leq \|f\|_p$, $(1 \leq p \leq \infty)$. *Contraction.*
- T_t *selfadjoint in* L^2.
- $T_t f \geq 0$ *if* $f \geq 0$. *Positivity.*
- $T_t 1 = 1$. *Markov.*

In these circumstances we say that the family $\{T_t\}_{t>0}$ is a diffusion semigroup.

E. Stein [23] proposed the use of of formulas related with the Gamma function in order to build a functional calculus for positive operators. Namely "fractional integrals" associated to a "Laplacian" (second order differential operator) can be given via the numerical formula
$\lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^\alpha e^{-\lambda t}\,\frac{dt}{t}$, $\lambda, \alpha > 0$. In fact if ϕ_k, is an eigenfunction of L with eigenvalue λ_k, we have

$$L^{-\alpha}\phi_k = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^\alpha e^{-tL}\phi_k \frac{dt}{t} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^\alpha e^{-t\lambda_k}\phi_k \frac{dt}{t} = \frac{1}{\lambda_k^\alpha}\phi_k.$$

In general, for positive L,

$$L^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha \, e^{-tL} f \, \frac{dt}{t},$$

where e^{-tL} is the heat semigroup associated to L.

As an illustration of the use of the formulas we compute the kernel of the fractional integral $(-\Delta)^{-\alpha/2}$ in $\mathbb{R}^n$, where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$. Let f be an smooth function with compact support, then

$$\begin{aligned}
(-\Delta)^{-\alpha/2} f(x) &= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2} \, e^{-t(-\Delta)} f(x) \, \frac{dt}{t} \\
&= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} f(y) \, dy t^{\alpha/2} \frac{dt}{t} \\
&= \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n} \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} t^{\alpha/2} \frac{dt}{t} f(y) \, dy \\
&= \frac{1}{(4\pi)^{n/2}} \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n} \int_0^\infty \frac{4^{(n-\alpha)/2}}{|x-y|^{n-\alpha}} u^{(n-\alpha)/2} \frac{du}{u} f(y) \, dy \\
&= \frac{1}{\pi^{n/2} 4^{\alpha/2} \Gamma(\alpha/2)} \Gamma\Big(\frac{n-\alpha}{2}\Big) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y) \, dy \\
&= c_{n,\alpha} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y) \, dy.
\end{aligned}$$

Observe that in the above calculation we don't use the Fourier transform.

Assume that we have a "Laplacian" with a factorization of the type $L = \partial^* \partial$. A natural "Riesz transform" associated to L should be $\partial (L)^{-1/2}$. Hence if $\{\phi_k\}_k$ is the family of orthogonal eigenfunctions we have

$$\begin{aligned}
&\int \big(\partial (L)^{-1/2} \phi_k\big)\big(\partial (L)^{-1/2} \phi_\ell\big) \, d\mu = \int \big(\partial^* \partial (L)^{-1/2} \phi_k\big)\big((L)^{-1/2} \phi_\ell\big) d\mu \\
&= \int \big(L (L)^{-1/2} \phi_k\big)\big((L)^{-1/2} \phi_\ell\big) \, d\mu = \int \big((L)^{1/2} \phi_k\big)\big((L)^{-1/2} \phi_\ell\big) \, d\mu \\
&= \lambda_k^{1/2} \, \lambda_\ell^{-1/2} \int \phi_k \, \phi_\ell \, d\mu.
\end{aligned}$$

This gives boundedness in $L^2(d\mu)$ of $Rf = \partial L^{-1/2}$.

What about the boundedness in L^p of Riesz transforms?. We shall make a quick discussion for the case of Hermite operator.

Let $\{H_k\}_{k \geq 0}$ be the family of 1-dimensional Hermite polynomials $H_k(x) = (-1)^k \dfrac{d^k}{dx^k}(e^{-x^2}) e^{x^2}$, $k = 0, 1 \ldots$, see [24]. Define the 1-dimensional Hermite functions $h_k(x) = (2^k k! \pi^{1/2})^{-1/2} H_k(x) e^{-x^2/2}$, $k = 0, 1, \ldots$, and the n-dimensional Hermite functions as $h_\alpha(x) = \Pi_{i=1}^n h_{\alpha_i}(x_i)$ $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. This is a complete orthonormal system in $L^2(\mathbb{R}^n)$.

Consider the operator

$$L = -\sum_i \frac{\partial^2}{\partial x_i^2} + |x|^2.$$

L satisfies $Lh_\alpha = (2|\alpha| + d)h_\alpha, \alpha = (\alpha_1, \ldots, \alpha_n)$, and it can be factorized as

$$L = -\sum_i \mathcal{A}_i\mathcal{A}_{-i} + \mathcal{A}_{-i}\mathcal{A}_i, \text{ with } \mathcal{A}_i = \frac{d}{dx} + x \text{ and } \mathcal{A}_i = \frac{d}{dx} + x.$$

In [24], S. Thangavelu defined the "Riesz" transforms associated to L as

$$\mathcal{R}_i = \mathcal{A}_i L^{-1/2}, 1 \leq |i| \leq n.$$

Thangavelu showed that they are bounded from $L^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$, for p in the range $1 < p < \infty$, and also of weak type $(1,1)$. In the paper [16] the authors gave a a new proof of Thangavelu's result with the advantage of showing that the L^p-boundedness, for $1 < p < \infty$, is a dimension free phenomenon. More precisely, the following theorem was proved:

Theorem 1.2. *Let p be in the range $1 < p < \infty$, then*

$$\Big\|\Big(\sum_{1\leq|i|\leq n} |\mathcal{R}_i f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)},$$

where C_p is a positive constant which depends only on p (and not on the dimension n).

The proof of Theorem 1.2 followed some ideas introduced by E. Stein in [21], and Gutiérrez in [15] using Littlewood–Paley g-functions.

2. Dimension free estimates for the maximal operator of the truncations

Along this section we shall use a technique of transference. It has its origins in, *inter alia*, the work of A. P. Calderón and A. Zygmund [7] on singular integrals and M. Cotlar [9] on the ergodic Hilbert transform. An early expository account of the idea of transference was given by Calderón [6] and this was followed by the monograph by R. R. Coifman and G. Weiss [8], who gave a comprehensive survey of the technique and its applications as the theory then stood.

Since the monograph of Coifman and Weiss appeared, several results involving the transfer of the boundedness of maximal operators and square functions associated to a family of operators have been obtained (see, e.g., [2], [1]) as well as an analogue that applies to representations on an arbitrary Banach space [5]. A vector valued version of the transference result established by Coifman and Weiss was developed in 2000 [4]. The precise statement is the following:

Theorem 2.1. *Let G be a locally compact abelian group, let X, Y be Banach spaces and let K be a function in $L^1_{\mathcal{L}(X,Y)}(G)$. Assume that there exist strongly continuous representations R and $\tilde{R}$ of the group G such that:*

(1) for every $u \in G$, we have $R_u \in \mathcal{L}(X,X)$ and $\tilde{R}_u \in \mathcal{L}(Y,Y)$;

(2) there exist constants c_1 and c_2 such that $\|R_u\|_{\mathcal{L}(X,X)} \leq c_1$ and $\|\tilde{R_u}\|_{\mathcal{L}(Y,Y)} \leq c_2,\ u \in G$;

(3) R and $\tilde{R}$ intertwine K, in the sense

$$K(u)R_v(x) = \tilde{R}_v K(u)(x),\ u,v \in G,\ x \in X.$$

We define the operator $T_K = \int_G K(u)R_{-u}du$. Then T_K is well defined as an element of $\mathcal{L}(X,Y)$ and

$$\|T_K\| \leq \inf_{1\leq p<\infty} (c_1 c_2 N_{p,X,Y}(K)),$$

where $N_{p,X,Y}(K)$ denotes the norm of the convolution operator defined by the kernel K from L^p_X into L^p_Y.

We shall also need the following

Definition 2.2. *Given Banach spaces B_1, B_2, and a function $k \in L^1(\mathbb{R})_{loc,\mathcal{L}(B_1,B_2)}$, we shall say that k is a Calderón–Zygmund kernel if there exists an operator K such that*

(i) *for some p_0, $1 < p_0 \leq \infty$, K maps $L^{p_0}_{B_1}(\mathbb{R})$ into $L^{p_0}_{B_2}(\mathbb{R})$;*

(ii) *if $\varphi \in L^\infty_{B_1}(\mathbb{R})$ and has compact support, then*

$$K\varphi(t) = \int_{\mathbb{R}} k(t-s)\varphi(s)ds, \quad t \notin \text{support of } \varphi;$$

.

(iii) *there exists a constant C such that*

(iii.1) $\|k(t)\| \leq C|t|^{-1}$ *and*

(iii.2) $\|k(t-s) - k(t)\| \leq C\frac{|s|}{|t|^2},\ 2|s| < |t|$.

Remark 2.3. *Given $\varepsilon > 0$ we denote by K_ε the operator obtained by truncating the kernel in the standard way, that is*

$$K_\varepsilon\varphi(t) = \int_{\{s:\varepsilon<|t-s|<1/\varepsilon\}} k(t-s)\varphi(s)ds.$$

It is well known that the operator K^* defined as $K^*\varphi(t) = \sup_\varepsilon \|K_\varepsilon\varphi(t)\|_{B_2}$ is bounded from $L^p_{B_1}(\mathbb{R},v)$ into $L^p(\mathbb{R},v)$, $1 < p < \infty$, for every weight v in the A_p class of Muckenhoupt. Moreover, the operator norm of K^* is majorized by a constant that depends only on the operator norm of K on $L^{p_0}_{B_1}(\mathbb{R})$, the constant C in (iii) and on the A_p constant of v.

We consider the operators $\partial(-\Delta)^{-\frac{1}{2}}$. These operators are defined for functions whose Fourier transforms have compact support in $\mathbb{R}_0$ by the formula

$$(\partial(-\Delta)^{-\frac{1}{2}}f)\hat{}(\xi) = (2\pi i\xi)|\xi|^{-1}\hat{f}(\xi).$$

Therefore they have bounded extensions to $L^2(\mathbb{R}^n)$. As we said before, the operator $(-\Delta)^{-\frac{1}{2}}$ can also be defined, in terms of the semigroup, as

$$(-\Delta)^{-\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty e^{t\Delta}\, t^{\frac{1}{2}}\frac{dt}{t}.$$

Therefore, by using the duality in $L^2(\mathbb{R}^n)$, the kernels associated, in the sense of Definition 2.2, with the operators $\partial(-\Delta)^{-\frac{1}{2}}$ as defined above can be computed. In fact, if f is a smooth compactly supported function, for all x outside the support of f we have

$$\begin{aligned}\partial_{x_j}(-\Delta)^{-\frac{1}{2}}f(x) &= \partial_{x_j}\frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n}\exp(-\frac{|x-y|^2}{4t})f(y)\,dy\,t^{\frac{1}{2}}\frac{dt}{t}\\ &= -\frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n}\frac{(x_j-y_j)}{2t}\exp(-\frac{|x-y|^2}{4t})f(y)\,dy\,t^{\frac{1}{2}}\frac{dt}{t}\\ &= -\frac{\Gamma(\frac{n+1}{2})}{2^{i\gamma}\pi^{n/2}\Gamma(\frac{1}{2})}\int_{\mathbb{R}^n}\frac{2(x_j-y_j)}{|x-y|^{n+1}}f(y)\,dy\\ &= -\frac{2^1\Gamma(\frac{n+1}{2})}{\omega_{n-1}\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}\int_{\mathbb{R}^n}\frac{(x_j-y_j)}{|x-y|^{n+1}}f(y)\,dy\\ &= \Delta_j * f(x),\end{aligned} \tag{2.4}$$

where $\omega_{n-1} = 2\pi^{n/2}/\Gamma(\frac{n}{2})$ is the surface area of the unit sphere in $\mathbb{R}^n$.

Given $\varepsilon > 0$, we denote

$$\Delta_{j,\varepsilon} * f(x) = -\frac{2^1\Gamma(\frac{n+1}{2})}{\omega_{n-1}\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}\int_{|x-y|>\varepsilon}\frac{(x_j-y_j)}{|x-y|^{n+1}}f(y)\,dy.$$

The dimension free theorem is the following:

Theorem 2.5. *Let p, $1 < p < \infty$, and α with $-1 < \alpha < p-1$. There exists a constant C_α, independent of n, such that*

$$\left\|\sup_\varepsilon\left(\sum_j|\Delta_{j,\varepsilon} * f|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n,|x|^\alpha dx)} \leq C_\alpha\|f\|_{L^p(\mathbb{R}^n,|x|^\alpha dx)}.$$

In order to prove this theorem, obtained in [12], we shall use some ideas from [11], [3] and [18]. But our intention is to present the proof of the result as an application of a weighted transference theory that can be developed in a vector valued setting (see [12] for a detailed discussion).

Given a σ-finite measure space $(X,\mathcal{F},\mu)$, an endomorphism of the σ-algebra $\mathcal{F}$ modulo null sets is a set function $\Phi : \mathcal{F} \to \mathcal{F}$ which satisfies

(i) $\Phi(\cup_n E_n) = \cup_n \Phi(E_n)$ for disjoint $E_n \in \mathcal{F}, n = 1, 2, \ldots$;
(ii) $\Phi(X \setminus E) = \Phi(X) \setminus \Phi(E)$ for all $E \in \mathcal{F}$;
(iii) given $E \in \mathcal{F}$, with $\mu(E) = 0$, then $\mu((\Phi)E) = 0$.

In these circumstances, Φ induces a unique positive and multiplicative linear operator, also denoted by Φ, on the space of (finite-valued or extended) measurable functions such that

$$\Phi(f_n) \to \Phi(f)\ \mu\text{ -a.e. whenever } 0 \leq f_n \to f,\ \mu\text{ -a.e.}$$

The action of Φ on simple functions is given by

$$\Phi(\sum_i c_i \chi_{E_i})(x) = \sum_i c_i \chi_{\Phi(E_i)}(x),\ c_i \in \mathbb{C}.$$

Given a Banach space B, Φ has an extension to the simple B-valued functions, also denoted by Φ, given by

$$\Phi(\sum_i \chi_{E_i} b_i) = \sum_i \chi_{\Phi(E_i)} b_i,\ (b_i \in X, E_i \in \mathcal{F}).$$

It is clear that, for $f : X \to B$ a simple function,

$$\|\Phi(f)(x)\|_B = \Phi(\|f\|_B)(x),\ (x \in X).$$

In other words, if Φ induces an operator T bounded in $L^p(\mu)$, then T has a bounded extension, also denoted by T, from $L^p_B(\mu)$ into $L^p_B(\mu)$ for any Banach space B. The action of T on $L^p(\mu) \otimes B$ is defined as

$$T(\sum_i \varphi_i b_i) = \sum_i T(\varphi_i) b_i,\ b_i \in B,\ \varphi_i \in L^p(\mu).$$

The norm of T on $L^p_B(\mu)$ equals the norm of T on $L^p(\mu)$.

Remark 2.6. *Throughout, we take $(X, \mathcal{F}, \mu)$ to be a σ-finite measure space and $\mathcal{T} = \{T^t : t \in \mathbb{R}\}$ a strongly continuous one-parameter group of positive invertible linear operators on $L^p = L^p(X, \mathcal{F}, \mu)$, for some fixed p in the range $1 < p < \infty$, such that, for each $t \in \mathbb{R}$, there exists a σ-endomorphism, Φ^t, with $T^t f = \Phi^t f$.*

From the group structure of $\mathcal{T}$, it follows that, for each $t \in \mathbb{R}$, there exists a positive function J_t such that

$$J_{t+s} = J_t \Phi^t J_s \text{ and } \int_X J_t \Phi^t f d\mu = \int_X f d\mu, \quad t, s \in \mathbb{R} \tag{2.7}$$

Using the properties of Bochner integration, we have

$$T^t(\int_K T^s f ds) = \int_K T^t(T^s f) ds,\ t \in \mathbb{R},$$

for all $f \in L^p(\mu)$ and all compact subsets K of $\mathbb{R}$.

Definition 2.8. *Let $(X, \mathcal{F}, \mu)$, $\mathcal{T}$ and fixed p in the range $1 < p < \infty$ be as in Remark 2.6, and let ω be a measurable function on X such that $\omega(x) > 0$, μ-almost everywhere. We shall say that ω is an ergodic A_p with respect to the group $\mathcal{T}$ if, for μ-almost all $x \in X$, the function $t \to J_t(x)\Phi^t(\omega)(x)$ is an A_p weight with an A_p-constant independent of x, where J_t and Φ^t are as in* (2.7).

We shall denote by $E_p(\mathcal{T})$ the class of ergodic A_p-weights associated with the group $\mathcal{T}$. Given a weight ω and a family $\mathcal{T}$ satisfying the hypothesis in Remark 2.6, we shall use the notation

$$\mathcal{T}\omega_x(t) = J_t(x)\Phi^t(\omega)(x).$$

In [12] a satisfactory weighted ergodic theory is developed; one of the outcomes obtained there is the following extrapolation result.

Theorem 2.9. *Let $\mathcal{T}$ be a family of operators satisfying the hypothesis in Remark 2.6 for every p in the range $1 < p < \infty$. Assume that K is a sublinear operator such that $\|Kf\|_{L^{p_0}(\omega d\mu)} \leq C_\omega \|f\|_{L^{p_0}(\omega d\mu)}$ for every $\omega \in E_{p_0}(\mathcal{T})$, where p_0 is fixed in the range $1 < p < \infty$ and the constant C_ω only depends on an $E_{p_0}(\mathcal{T})$-constant for ω. Then K is bounded from $L^p(\omega d\mu)$ into $L^p(\omega d\mu)$ for every p, $1 < p < \infty$, and every $\omega \in E_p(\mathcal{T})$.*

We now state the transference theorem, whose proof can be found in [12], see also [13]. Recall that T^t has a natural extension to $L^p_{B_1}(X, d\mu)$, also denoted by T^t.

Theorem 2.10. *Let $1 < p < \infty$ and let $\mathcal{T}$ be a group of operators satisfying the hypothesis in Remark 2.6. Let B_1, B_2, be Banach spaces and K a Calderón–Zygmund operator with kernel k as in Definition 2.2. Given a finite set $\mathcal{J} \subset (0, \infty)$, we define the operator $C_K^{\mathcal{J}}$ on $L^p_{B_1}(X, \mu)$ by*

$$C_K^{\mathcal{J}} f(x) = \max_{\varepsilon \in \mathcal{J}} \| \int_{\{\varepsilon < |s| < 1/\varepsilon\}} k(s) T^{-s} f(x) ds \|_{B_2}.$$

Then

$$\sup_{\{\mathcal{J} : \mathcal{J} \text{ finite } \subset (0,\infty)\}} \|C_K^{\mathcal{J}} f\|_{L^p(X,\omega)} \leq N_p(K, \mathcal{T}\omega) \|f\|_{L^p_{B_1}(X,\omega)}$$

for every $\omega \in E_p(\mathcal{T})$. Here $N_p(K, \mathcal{T}\omega)$ denotes an essential bound relative to x of the operator-norm of K^* as a bounded operator from $L^p_{B_1}(\mathbb{R}, \mathcal{T}\omega_x)$ into $L^p(\mathbb{R}, \mathcal{T}\omega_x)$, where $\mathcal{T}\omega_x(t)$ is defined in 2.

We observe that $\mathcal{T}\omega_x(\cdot) \in A_p$ with an A_p constant independent of x, since $\omega \in E_p(\mathcal{T})$, and hence such essential bounds exist.

Let k be a Calderón–Zygmund kernel with the corresponding operator K. We consider the unit sphere Σ_{n-1} of $\mathbb{R}^n$ endowed with the rotationally invariant measure $d\sigma$ normalized so that $\int_{\Sigma_{n-1}} d\sigma = 1$. Given a fixed $y' \in \Sigma_{n-1}$ we consider the one-parameter group of operators $\mathcal{T}_{y'} = \{\Phi^t_{y'}\}_t$, where

$$\Phi^t_{y'}(f)(x) = f(x + ty'),\ x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

Clearly, $\|\Phi^t_{y'}(f)\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}$. Therefore, if

$$C_{K,\varepsilon,y'} = \int_{\{\varepsilon < |s| < 1/\varepsilon\}} k(s) \Phi^{-s}_{y'} ds, \tag{2.11}$$

then by Theorem 2.10

$$\|\{C_{K,\varepsilon,y'} f\}_{\varepsilon \in \mathcal{J}}\|_{L^p_{\ell^\infty(\mathcal{J})}(\mathbb{R}^n,\omega)} \leq N_p(K, \mathcal{T}_{y'}\omega) \|f\|_{L^p(\mathbb{R}^n,\omega)} \tag{2.12}$$

for every finite subset $\mathcal{J}$ of $(0, \infty)$ and every $\omega \in E_p(\mathcal{T}_{y'})$, where $1 < p < \infty$.

Let P be the projection of the space $L^2(d\sigma)$ into the subspace H of $L^2(d\sigma)$ generated by the functions $y'_1, \ldots, y'_n$.

Lemma 2.13. *With the notations in* (2.11)*, we have*

$$P(C_{K,\varepsilon,\cdot}f(x))(y') = \sum_{j=1}^{n} C^j_{K,\varepsilon} f(x) Y_j(y'),\ \ f \in L^\infty,$$

where

$$C^j_{K,\varepsilon} f(x) = \frac{1}{\omega_{n-1}} \int_{\{z\in\mathbb{R}^n : \varepsilon<|z|<\frac{1}{\varepsilon}\}} \frac{k(|z|)-k(-|z|)}{|z|^{n-1}} f(x-z) Y_j(\frac{z}{|z|}) dz,\ j=1,\dots,n$$

and $\{Y_j\}_{j=1}^n$ *are the functions* $Y_j(y') = n^{1/2} y'_j$ *for* $y' \in \Sigma_{n-1}$.

Proof. As P is a projection and $Y_1,\dots,Y_n$ are orthonormal in $L^2(\Sigma_{n-1}, d\sigma)$, we have

$$P(C_{K,\varepsilon,\cdot}f(x))(y') = \sum_j c_j(x) Y_j(y'),$$

where

$$c_j(x) = \int_{\Sigma_{n-1}} C_{K,\varepsilon,y'} f(x) Y_j(y') d\sigma(y').$$

By using polar coordinates and the fact that the $Y_j's$ are odd functions, the proof can be finished.

□

Theorem 2.14. *Let K be a Calderón–Zygmund operator on $\mathbb{R}$ with associated kernel k as in 2.2. Let $1 < p < \infty$, assume that ω is a weight in $\mathbb{R}^n$ such that the function $t \to \Phi^t_{y'}\omega(x)$ is a weight in $A_p(\mathbb{R})$ with an A_p-constant independent of y' and x. Then there exists a constant C such that*

$$\|\{ (\sum_{j=1}^n |C^j_{K,\varepsilon} f|^2)^{1/2} \}_{\varepsilon\in\mathcal{J}} \|_{L^p_{\ell^\infty(\mathcal{J})}(\mathbb{R}^n,\omega)} \le C\|f\|_{L^p(\mathbb{R}^n,\omega)} \tag{2.15}$$

for every finite subset $\mathcal{J}$ in $(0,\infty)$. Moreover the constant C can be taken to be an upper bound for the norm of operators of the form $K^* : L^p(\mathbb{R}, v) \to L^p(\mathbb{R}, v)$, where $v(t) = \Phi^t_{y'}\omega(x)$ for some y' and x (see Remark 2.3).

Proof. We observe that by using Theorem 2.9 it is enough to prove inequality (2.15) for some p, $1 < p < \infty$. We shall prove it for $p = 2$. In fact, using orthogonality and the representation formula for P' in Lemma 2.13, we have

$$
\begin{aligned}
&\left\| \left\{ \left(\sum_{j=1}^{n} |C^j_{K,\varepsilon} f|^2 \right)^{1/2} \right\}_{\varepsilon\in\mathcal{J}} \right\|_{L^2_{\ell^\infty(\mathcal{J})}(\mathbb{R}^n,\omega)} \\
&\quad = \left\| \left\{ \left(\int_{\Sigma_{n-1}} \left| \sum_{j=1}^{n} C^j_{K,\varepsilon} f Y_j(y') \right|^2 d\sigma(y') \right)^{1/2} \right\}_{\varepsilon\in\mathcal{J}} \right\|_{L^2_{\ell^\infty(\mathcal{J})}(\mathbb{R}^n,\omega)} \\
&\quad = \left\| \left\{ \left(\int_{\Sigma_{n-1}} |P(C_{K,\varepsilon,\cdot} f(\cdot))(y')|^2 d\sigma(y') \right)^{1/2} \right\}_{\varepsilon\in\mathcal{J}} \right\|_{L^2_{\ell^\infty(\mathcal{J})}(\mathbb{R}^n,\omega)} \\
&\quad \le \left\| \left\{ \left(\int_{\Sigma_{n-1}} |C_{K,\varepsilon,y'} f(\cdot)|^2 d\sigma(y') \right)^{1/2} \right\}_{\varepsilon\in\mathcal{J}} \right\|_{L^2_{\ell^\infty(\mathcal{J})}(\mathbb{R}^n,\omega)} \\
&\quad \le \left(\int_{\Sigma_{n-1}} \left(\| \{ C_{K,\varepsilon,y'} f(\cdot) \}_{\varepsilon\in\mathcal{J}} \|_{L^2_{\ell^\infty(\mathcal{J})}(\mathbb{R}^n,\omega)} \right)^2 d\sigma(y') \right)^{1/2} \\
&\quad \le \left(\int_{\Sigma_{n-1}} N_2(K, \mathcal{T}_{y'}\omega)^2 \|f\|^2_{L^2(\mathbb{R}^n,\omega)} \, d\sigma(y') \right)^{1/2} \\
&\quad \le C \|f\|_{L^2(\mathbb{R}^n,\omega)},
\end{aligned}
$$

where in the penultimate inequality we have used (2.12). □

Corollary 2.16. *Let $1 < p < \infty$ and let $-1 < \alpha < p-1$. Let K be a Calderón–Zygmund operator on $\mathbb{R}$ with associated kernel k as in 2.2 and consider the operators $C^j_{K,\varepsilon}$ defined in Lemma 2.13. Then there exists a constant $C_{\alpha,p}$ such that*

$$
\int_{\mathbb{R}^n} \sup_{\varepsilon>0} \Big(\sum_{j=1}^{n} |C^j_{K,\varepsilon} f(x)|^2 \Big)^{p/2} |x|^\alpha \, dx \le C_{\alpha,p} \int_{\mathbb{R}^n} |f(x)|^p |x|^\alpha \, dx.
$$

Proof. In order to use Theorem 2.14, it will be enough to show that, given $x \in \mathbb{R}^n$ and $y' \in \Sigma_{n-1}$, the function $t \to |x + ty'|^\alpha$ is an A_p-weight on $\mathbb{R}$, with an A_p-constant independent of x and y'.

Fix $x \in \mathbb{R}^n, y' \in \Sigma_{n-1}$ and decompose x as $x = x_1 + t_0 y'$, with $x_1 \perp y'$. Then, as $|y'| = 1$, we have $|x + ty'| = (|x_1|^2 + |t_0 + t|^2)^{1/2} \sim |x_1| + |t_0 + t|$. Therefore $|x+ty'|^\alpha \sim |x_1|^\alpha + |t_0+t|^\alpha$. Hence if M is the Hardy–Littlewood maximal operator and we denote by φ_s the translate function $\varphi_s(t) = \varphi(t-s)$, by using the translation properties of Lebesgue measure and the fact that $|t|^\alpha$ is a A_p-weight, we have

$$\begin{aligned}\int_{\mathbb{R}} |M\varphi(t)|^p(|x_1|^\alpha + |t_0+t|^\alpha)dt &= \int_{\mathbb{R}} |M\varphi(t)|^p |x_1|^\alpha dt + \int_{\mathbb{R}} |M\varphi(t)|^p|t_0+t|^\alpha dt \\ &= |x_1|^\alpha \int_{\mathbb{R}} |M\varphi(t)|^p dt + \int_{\mathbb{R}} |M\varphi_{t_0}(t)|^p|t|^\alpha dt \\ &\le |x_1|^\alpha C_p \int_{\mathbb{R}} |\varphi(t)|^p dt + A_p(|t|^\alpha)\int_{\mathbb{R}} |\varphi_{t_0}(t)|^p |t|^\alpha dt \\ &\le (C_p + A_p(|t|^\alpha)) \int_{\mathbb{R}} |\varphi(t)|^p(|x_1|^\alpha + |t_0+t|^\alpha)dt.\end{aligned}$$

It follows that $|x_1|^\alpha + |t_0+t|^\alpha$, and hence $|x+ty'|^\alpha$, is an A_p-weight with an A_p-constant on $\mathbb{R}$ independent of x and y'. □

Proof of Theorem 2.5. We consider the Calderón–Zygmund operators on $\mathbb{R}$ given by the Calderón–Zygmund kernels $k_0(t) = |t|^{-1+i\gamma}$ with $\gamma \neq 0$ and $k_1(t) = t^{-1}$ (see [22, Ch. II]). Therefore with this notation we have in Lemma 2.13

$$C^j_{K_1,\varepsilon} f(x) = \frac{2n^{1/2}}{\omega_{n-1}} \int_{\{z\in\mathbb{R}^n : \varepsilon<|z|<\frac{1}{\varepsilon}\}} \frac{1}{|z|} f(x-z)\frac{z_j}{|z|^n}dz,$$

and so, by (2.4) with $\gamma = 0$,

$$C^j_{K_1,\varepsilon} f(x) = -\kappa_n \Delta_{j,\varepsilon} * f(x),$$

where $\kappa_n = \frac{n^{1/2}\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})}\Delta_{j,\varepsilon}$. As before, Stirling's formula gives $|\kappa_n| \sim C$ and therefore the case $m = 1$ in the theorem follows from Corollary 2.16.

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