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THE LIMIT CASE IN A NONLOCAL p-LAPLACIAN EQUATION WITH DYNAMICAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper we deal with the limit as $p \to \infty$ for the nonlocal analogous to the p-Laplacian evolution with dynamic boundary conditions. Our main result demonstrates this limit in both the elliptic and parabolic cases. We are interested in smooth and singular kernels and show the existence and uniqueness of a limit solution. We obtain that the limit solution of the elliptic problem turns out to be also a viscosity solution of a corresponding problem. We prove that the natural energy functionals associated with this problem converge, in the sense of Mosco, to a limit functional and therefore we obtain convergence of solutions to the evolution problems in the parabolic case. For the limit problem, we provide examples of explicit solutions for some particular data.

1. Introduction

Our main purpose in this paper is to study a nonlocal diffusion equation obtained as the limit as $p \to \infty$ to the p-Laplacian with dynamical boundary conditions, that is, we look for the limit as $p \to \infty$ of the solutions to the following problem (P1):

$$\begin{cases} 0 = \displaystyle\int_{\Omega_r \cup \Gamma_r} J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy, & x \in \Omega_r,\ t>0; \\ \dfrac{\partial u}{\partial t}(x,t) = \displaystyle\int_{\Omega_r} J(x-y)|u(y,t)-u(x,t)|^{p-2} & \\ \qquad\qquad \times(u(y,t)-u(x,t))\,dy + f(x,t), & x \in \Gamma_r,\ t>0; \\ u(x,0) = u_0(x), & x \in \Gamma_r, \end{cases} \tag{1.1}$$

where $\hat{\Omega}$ is a smooth bounded domain and inside this domain a narrow strip $\Gamma_r = \{x \in \hat{\Omega} : \operatorname{dist}(x, \partial\hat{\Omega}) \le r\}$, with $r \le R$, $\Omega_r = \hat{\Omega} \setminus \Gamma_r$. We consider $1 < p < \infty$ and the limit $p \to \infty$. We assume that the non-singular kernel $J : \mathbb{R}^n \to \mathbb{R}$ is nonnegative, continuous, radially symmetric, decreasing, and compactly supported

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(let $\operatorname{supp}(J) = B_R(0)$) with $\int J(w)\,dw = 1$. We also analyze the case in which the kernel J can be singular.

First of all, we introduce the linear form of nonlocal equations,

$$\frac{\partial u}{\partial t}(x,t) = \int J(x-y)|u(y,t)-u(x,t)|(u(y,t)-u(x,t))\,dy.$$

These types of equations have been widely used in the modeling of diffusion processes. More precisely, as stated in [20], if $u(x,t)$ is thought of as a density at the point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int J(y-x)u(y,t)\,dy$ is the rate at which individuals are arriving at position x from all other places and $-\int J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to travel to all other sites. Then these nonlocal equations give that the change in time of the density of individuals at x at time t is just the balance between arriving to/leaving from x at time t (see [4, 17, 18, 20]). According to this probabilistic interpretation of the nonlocal terms, as noted in [9], we can regard (1.1) as a model for the following situation: particles leave or arrive from $x \in \Omega_r$ to $y \in \Omega_r \cup \Gamma_r$ in very fast time scales giving rise to an "elliptic" equation inside Ω_r (notice that t is only a parameter in the first equation that appears in (P1)). On the other hand, individuals arrive to or leave from $x \in \Gamma_r$ from other sites $y \in \Omega_r$ in the slow time scale. This gives the second equation in (P1) in which a time derivative appears. The existence and uniqueness of mild and strong solutions of nonlocal nonlinear diffusion problems of p-Laplacian type with nonlinear boundary conditions posed in metric random walk spaces were studied in [23, 27]. For recent works on nonlocal diffusion, see [3, 7, 8, 14, 16, 17, 20, 21, 28, 29].

Concerning limits as $p \to \infty$, one of the first papers that studies this kind of problems is [10]. Let u_p denote the solution to the problem

$$\begin{cases} -\Delta_p u_p = f & \text{in } \Omega, \\ u_p = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $f \in C(\Omega)$ and $f > 0$. Then, from [10], we have that u_p converges uniformly as $p \to \infty$ to the distance to the boundary, that is, $\lim_{p\to\infty} u_p(x) = u_\infty(x) = d(x,\partial\Omega)$ for $x \in \overline{\Omega}$.

In [5] and [19], the limiting behavior as $p \to \infty$ of solutions to the quasilinear parabolic problem

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) - \Delta_p v(x,t) = f(x,t) & \text{in } (0,T) \times \mathbb{R}^N, \\ v(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

was studied. In [5], assuming that u_0 is a Lipschitz function with compact support satisfying $|\nabla u_0| \le 1$, it is proved that $v_p \to v_\infty$ and the limit function v_∞ satisfies

$$f(x,t) - \frac{\partial v_\infty}{\partial t}(x,t) \in \partial F_\infty(v_\infty(x,t)),$$

with

$$F_\infty(v) = \begin{cases} 0 & \text{if } |\nabla v| \le 1, \\ +\infty & \text{in other case.} \end{cases}$$

The limit problem explains the movement of a sandpile ($v_\infty(t,x)$ describes the amount of the sand at the point x and time t), the main assumption being that the sandpile is stable when the slope is less than or equal to one and unstable if not.

In [26], we looked for the nonlinear diffusion equation obtained as the limit as $p \to \infty$ to the p-Laplacian with dynamical boundary conditions, that is, we looked for the limit as $p \to \infty$ of the solutions to the following problem:

$$\begin{cases} 0 = \Delta_p u(x,t), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial t}(x,t) + |\nabla u|^{p-2} \dfrac{\partial u}{\partial \eta}(x,t) = f(x,t), & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \partial\Omega. \end{cases}$$

We proved that the natural energy functionals associated with this problem converge in the sense of Mosco to a limit functional and therefore we obtain convergence of the solutions to the evolution problems.

In [9], the authors deal both with smooth and singular kernels and show the existence and uniqueness of solutions and study their asymptotic behavior as t goes to infinity for (P1) with homogeneous dynamic boundary conditions.

In [1], the authors study on the nonlocal ∞-Laplacian type diffusion equation obtained as the limit as $p \to \infty$ to the nonlocal analogous to the p-Laplacian evolution,

$$\begin{cases} u_t(t,x) = \displaystyle\int_{\mathbb{R}^N} J(x-y)|u(t,y) - u(t,x)|^{p-2}(u(t,y) - u(t,x))\,dy + f(x,t) \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N,\ t > 0. \end{cases}$$

They prove existence and uniqueness of a limit solution that satisfies an equation governed by the subdifferential of a convex energy functional associated to the indicator function of the set

$$K = \{u \in L^2(\mathbb{R}^N) : |u(x) - u(y)| \le 1 \text{ when } x - y \in \operatorname{supp}(J)\}.$$

In [2], the authors study the nonlocal p-Laplacian Dirichlet problem

$$\begin{cases} u_t(t,x) = \displaystyle\int_{\Omega} J(x-y)|u(t,y) - u(t,x)|^{p-2}(u(t,y) - u(t,x))\,dy + f(x,t), \\ u(x,t) = \varphi(x), \quad x \in \Omega_J \setminus \overline{\Omega},\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases}$$

where $\Omega_J = \Omega + \operatorname{supp}(J)$ and φ is a given function $\varphi : \Omega_J \setminus \overline{\Omega} \to \mathbb{R}$. In this paper, the authors prove the existence and uniqueness of strong solutions for the nonlocal p-Laplacian problem with Dirichlet boundary conditions for $p > 1$ and they show that this model approaches the local p-Laplacian evolution equation with Dirichlet boundary conditions. They study the Dirichlet problem for the nonlocal total variation flow, proving convergence to the local model when the problem is rescaled appropriately as well. Lastly they study the case $p = \infty$, obtaining a model for sandpiles with Dirichlet boundary conditions.

We have been inspired by the works [2, 1], closely related to the present work, where the Neumann and Dirichlet boundary value problems and their limits as p goes to infinity are considered. The difference here is that we are now considering dynamic boundary conditions with the nonhomogeneous case.

The rest of the paper is organized as follows. In Section 2, we recall some useful results that will be used in the proofs of theorems, among them some technical tools from convex analysis. In Section 3, we consider the limit $p \to \infty$ in the elliptic case with smooth and with singular kernels; also, in the case of singular kernels, we obtain that the limit solution of the elliptic problem turns out to be also a viscosity solution of a corresponding problem. In Section 4, we prove that the natural energy functionals associated with (P1) converge in the sense of Mosco to a limit functional and therefore we obtain convergence of the solutions to the evolution problems. In Section 5, for the limit problem we provide examples of explicit solutions for some particular data.

2. Preliminaries

In this section, we collect some preliminaries and notations that will be used in the paper. We refer the reader to [4, 6, 12, 24, 25].

First, we recall the definition of Mosco convergence. If X is a metric space, and $\{A_n\}$ is a sequence of subsets of X, we define

$$\liminf_{n\to\infty} A_n := \big\{x \in X : \text{there exists } x_n \in A_n,\, x_n \to x\big\}$$

and

$$\limsup_{n\to\infty} A_n := \big\{x \in X : \text{there exists } x_{n_k} \in A_{n_k},\, x_{n_k} \to x\big\}.$$

If X is a normed space, we denote by s-lim and w-lim the above limits associated, respectively, to the strong and the weak topology of X.

Definition 2.1. Let H be a Hilbert space. Given $\Psi_n, \Psi : H \to (-\infty, +\infty]$ convex, lower-semicontinuous functionals, we say that Ψ_n converges to Ψ in the sense of Mosco if

$$w\text{-}\limsup_{n\to\infty} \operatorname{Epi}(\Psi_n) \subset \operatorname{Epi}(\Psi) \subset s\text{-}\liminf_{n\to\infty} \operatorname{Epi}(\Psi_n), \tag{2.1}$$

where $\operatorname{Epi}(\Psi_n)$ and $\operatorname{Epi}(\Psi)$ denote the epigraphs of the functionals Ψ_n and Ψ, defined by

$$\operatorname{Epi}(\Psi_n) := \big\{(u,\lambda) \in L^2(\mathbb{R}^N) \times \mathbb{R} : \lambda \geq \Psi_n(u)\big\}$$

and

$$\operatorname{Epi}(\Psi) := \big\{(u,\lambda) \in L^2(\mathbb{R}^N) \times \mathbb{R} : \lambda \geq \Psi(u)\big\}.$$

Remark 2.2. We note that (2.1) is equivalent to the requirement that the following two conditions are simultaneously satisfied:

- for all $u \in D(\Psi)$ there exists $u_n \in D(\Psi_n)$ such that $u_n \to u$ and $\Psi(u) \geq \limsup_{n\to\infty} \Psi_n(u_n)$;
- for every subsequence $\{n_k\}$, $\Psi(u) \leq \liminf_k \Psi_{n_k}(u_k)$ whenever $u_k \rightharpoonup u$.

Here $D(\Psi) := \{u \in H : \Psi(u) < \infty\}$ and $D(\Psi_n) := \{u \in H : \Psi_n(u) < \infty\}$ denote the domains of Ψ and Ψ_n, respectively.

If H is a real Hilbert space with inner product $(\cdot,\cdot)$ and $\Psi : H \to (-\infty, +\infty]$ is convex, then the subdifferential of Ψ is defined as the multivalued operator $\partial\Psi$ given by

$$v \in \partial\Psi(u) \iff \Psi(w) - \Psi(u) \geq (v, w-u) \text{ for all } w \in H.$$

Recall that the epigraph of Ψ is defined by

$$\mathrm{Epi}(\Psi) = \big\{(u,\lambda) \in H \times \mathbb{R} : \lambda \geq \Psi(u)\big\}.$$

Given K a closed convex subset of H, we define the indicator function of K by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

Then the subdifferential is characterized by

$$v \in \partial I_K(u) \iff u \in K \text{ and } (v, w-u) \leq 0 \text{ for all } w \in K.$$

When the convex functional $\Psi : H \to (-\infty, +\infty]$ is proper, lower-semicontinuous, and such that $\min \Psi = 0$, it is well known (see [11]) that the abstract Cauchy problem

$$\begin{cases} u_t + \partial\Psi(u) \ni f & \text{a.e. } t \in (0,T), \\ u(0) = u_0, \end{cases}$$

has a unique solution for any $f \in L^1(0,T;H)$ and $u_0 \in \overline{D(\partial\Psi)}$.

The Mosco convergence is a very useful tool to study the convergence of solutions of parabolic problems. The following theorem is a consequence of results in [6], [13].

Theorem 2.3. *Let $\Psi_n, \Psi : H \to (-\infty, +\infty]$ be convex and lower semicontinuous functionals. Then, the following statements are equivalent:*

(i) *Ψ_n converges to Ψ in the sense of Mosco.*
(ii) *$(I + \lambda\partial\Psi_n)^{-1}u \to (I + \lambda\partial\Psi)^{-1}u$ for all $\lambda > 0$, $u \in H$.*

Moreover, either one of the above conditions, (i) *or* (ii)*, implies that*

(iii) *for every $u_0 \in \overline{D(\partial\Psi)}$ and $u_{0,n} \in \overline{D(\partial\Psi_n)}$ such that $u_{0,n} \to u_0$, and for every $f_n, f \in L^1(0,T;H)$ with $f_n \to f$, if $u_n(t)$, $u(t)$ are solutions of the abstract Cauchy problems*

$$\begin{cases} (u_n)_t + \partial\Psi_n(u_n) \ni f_n & a.e.\ t \in (0,T) \\ u_n(0) = u_{0,n}, \end{cases}$$

and

$$\begin{cases} u_t + \partial\Psi(u) \ni f \quad a.e.\ t \in (0,T) \\ u(0) = u_0, \end{cases}$$

respectively, then

$$u_n \to u \quad in\ C([0,T] : H).$$

3. Elliptic case

As a first step, we consider the problem (P1) in the elliptic case, that is, drop the time dependence and take the limit as $p \to \infty$ in the following problem (P1s):

$$\begin{cases} 0 = \displaystyle\int_{\Omega_r \cup \Gamma_r} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, & x \in \Omega_r, \\ f(x) = \displaystyle\int_{\Omega_r} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, & x \in \Gamma_r. \end{cases}$$

Theorem 3.1. *Assume that $\int_{\Gamma_r} f(x)\,dx = 0$. For p fixed there exists a unique solution u_p (that depends on p) of* (P1s) *such that*

$$\int_{\Gamma_r} u_p(x) = 0.$$

For the proof of this theorem, we use variational arguments. Let us define the functional $J_p(u)$ as follows:

$$J_p(u) = \frac{1}{2p}\iint_H J(x-y)|u(y)-u(x)|^p\,dy\,dx - \int_{\Gamma_r} fu,$$

with

$$H = (\Omega_r \cup \Gamma_r) \times (\Omega_r \cup \Gamma_r) \setminus \Gamma_r \times \Gamma_r.$$

We will prove some properties of $J_p(u)$ in the following lemmas.

Lemma 3.2. *There is a unique solution u_p to the problem*

$$\min_{u \in B_p} J_p(u)$$

in the set

$$B_p = \Big\{ u \in L^p(\hat{\Omega}) : \int_{\Gamma_r} u(x) = 0 \Big\}.$$

Proof. To find the existence of such a minimum we need a Poincaré-type inequality: there is a constant c such that

$$\iint_H J(x-y)|u(y)-u(x)|^p\,dy\,dx \geq c\int_{\Gamma_r} |u|^p \tag{3.1}$$

for every $u \in B_p$. This inequality is proved in [9, Theorem 2.5]. By using the above inequality, along with Hölder's and Young's inequalities, we obtain the following:

$$\begin{aligned} J_p(u) &= \frac{1}{2p} \iint_H J(x-y)|u(y)-u(x)|^p \, dy \, dx - \int_{\Gamma_r} fu \\ &\geq \frac{c}{2p} \int_{\Gamma_r} |u|^p - c(\varepsilon)\|f\|^{p'}_{L_{p'}(\Gamma_r)} - \varepsilon \|u\|^p_{L_p(\Gamma_r)}. \end{aligned}$$

Taking $\varepsilon \leq \frac{c}{2p}$, we get

$$J_p(u) \geq -c(\varepsilon)\|f\|^{p'}_{L_{p'}(\Gamma_r)}.$$

This means that we can obtain the existence of the infimum of $J_p(u)$ in B_p. Then there exists $\{u_n\} \in B_p$ such that

$$J_p(u_n) \to \inf_{B_p} J_p(u) > -\infty.$$

Hence we have that $\{J_p(u_n)\}$ is bounded. Then

$$\frac{1}{2p} \iint_H J(x-y)|u_n(y)-u_n(x)|^p \, dy \, dx - \int_{\Gamma_r} fu_n \leq C.$$

Taking into account (3.1) and using Hölder's inequality we have

$$\frac{c}{2p} \int_{\Gamma_r} |u_n|^p \leq C + \int_{\Gamma_r} fu_n \, dx \leq C + \|f\|_{L_{p'}(\Gamma_r)} \|u_n\|_{L_p(\Gamma_r)}.$$

Using Young's inequality we get

$$\frac{c}{2p} \int_{\Gamma_r} |u_n|^p \leq C + c(\varepsilon) \frac{\|f\|^{p'}_{L_{p'}(\Gamma_r)}}{p'} + \varepsilon \frac{\|u_n\|^p_{L_p(\Gamma_r)}}{p}.$$

Then

$$\left(\frac{\frac{c}{2} - \varepsilon}{p} \right)^{\frac{1}{p}} \|u_n\|_{L_p(\Gamma_r)} \leq \left[C + \frac{c(\varepsilon)}{p'} \|f\|^{p'}_{L_{p'}(\Gamma_r)} \right]^{\frac{1}{p}}.$$

Hence we obtain that u_n is bounded in $L_p(\Gamma_r)$ independently of p. Hence for some subsequence $u_{n_j} \subset u_n$ and $u_p \in L_p(\Gamma_r)$ we have

$$u_{n_j} \rightharpoonup u_p \quad \text{in } L_p(\Gamma_r).$$

On the other hand, by weak convergence,

$$\begin{aligned} &\frac{1}{2p} \iint_H J(x-y)|u_p(y)-u_p(x)|^p \, dy \, dx \\ &\qquad\qquad \leq \frac{1}{2p} \liminf_{n_j \to \infty} \iint_H J(x-y)|u_{n_j}(y)-u_{n_j}(x)|^p \, dy \, dx \end{aligned}$$

and

$$-\int_{\Gamma_r} u_{n_j} f \to -\int_{\Gamma_r} u_p f.$$

Combining the previous two limits we obtain

$$\frac{1}{2p}\iint_H J(x-y)|u_p(y)-u_p(x)|^p\,dy\,dx-\int_{\Gamma_r}u_pf \\ \leq \liminf_{n_j\to\infty}\frac{1}{2p}\iint_H J(x-y)|u_{n_j}(y)-u_{n_j}(x)|^p\,dy\,dx-\int_{\Gamma_r}u_{n_j}f.$$

Hence we can write

$$J_p(u_p)\leq\liminf_{n_j\to\infty}J_p(u_{n_j})=\inf_{B_p}J_p(u),$$

which implies u_p is a minimizer of $J_p(u)$ in B_p. The uniqueness follows by the strict convexity of the functional J_p. Let us assume that we have two minimizers $u_1\neq u_2\in B_p$, $w:=\frac{u_1+u_2}{2}\in B_p$ and

$$J_p(w)<\frac{1}{2}(J_p(u_1)+J_p(u_2))=\inf_{B_p}J_p(u).$$

This implies that $J_p(w)<\inf_{B_p}J_p(u)$, which contradicts that u_p is the minimizer of $J_p(u)$ in B_p. □

Now we are ready to prove our existence and uniqueness result.

Proof of Theorem 3.1. As an immediate consequence of our previous results, the unique minimizer $u_p\in B_p$ is the unique solution of (P1s). □

Now we want to obtain an estimate on u_p.

Lemma 3.3. *Let u_p be the minimizer of $J_p(u)$ in B_p. Then there exists a constant C independent of p such that*

$$\|u_p\|_{L_p(\Gamma_r)}\leq C$$

for p large enough.

Proof. Let us take $u_0\in L_\infty(\hat\Omega)$ such that $|u_0(x)-u_0(y)|\leq 1$ for almost every $x,y\in H$. We have

$$J_p(u_0)=\frac{1}{2p}\iint_H J(x-y)|u_0(y)-u_0(x)|^p\,dy\,dx-\int_{\Gamma_r}fu_0 \\ \leq\frac{1}{2p}\iint_H J(x-y)\,dy\,dx-\int_{\Gamma_r}fu_0=C(u_0).$$

Since u_p is the minimizer of $J_p(u)$, we have

$$\frac{1}{2p}\iint_H J(x-y)|u_p(y)-u_p(x)|^p\,dy\,dx\leq C(u_0)+\int_{\Gamma_r}fu_p.$$

Taking into account (3.1) and using Hölder's inequality we have

$$\frac{c}{2p}\int_{\Gamma_r}|u_p|^p\leq C(u_0)+\int_{\Gamma_r}fu_p\,dx\leq C(u_0)+\|f\|_{L_{p'}(\Gamma_r)}\|u_p\|_{L_p(\Gamma_r)}.$$

Using Young's inequality we get

$$\left(\frac{\frac{c}{2}-\varepsilon}{p}\right)^{\frac{1}{p}} \|u_p\|_{L_p(\Gamma_r)} \le \left[C(u_0) + \frac{c(\varepsilon)}{p'}\|f\|^{p'}_{L_{p'}(\Gamma_r)}\right]^{\frac{1}{p}}.$$

Hence we obtain that u_p is bounded in $L_p(\Gamma_r)$ independently of p:

$$\|u_p\|_{L_p(\Gamma_r)} \le C$$

for p large enough. □

We obtain the following trivial consequence of Lemma 3.3.

Remark 3.4. We have that for the unique minimizer u_p there exists some constant C independent of p such that

$$\frac{1}{2p}\iint_H J(x-y)|u_p(y)-u_p(x)|^p\,dy\,dx \le C.$$

As a consequence of Lemma 3.3 we can extract a subsequence $\{u_{p_j}\} \subset u_p$ and $u_{p_j} \rightharpoonup u_\infty$ weakly in $L_p(\Gamma_r)$.

Lemma 3.5. *Let u_∞ be a limit of u_p (along a subsequence if necessary); then there exists a constant $C(q)$ such that*

$$\lim_{q\to\infty} C(q) = 1$$

and

$$\left(\iint_H J(x-y)|u_\infty(y)-u_\infty(x)|^q\,dx\,dy\right)^{1/q} \le C(q).$$

Proof. We have

$$\begin{aligned}\left(\iint_H J(x-y)|u_{p_j}(y)-u_{p_j}(x)|^q\,dx\,dy\right)^{1/q}\\ = \left(\iint_H (J(x-y))^{1-\frac{q}{p_j}}\left[(J(x-y))^{\frac{q}{p_j}}|u_{p_j}(y)-u_{p_j}(x)|^q\right]\right)^{\frac{1}{q}}.\end{aligned}$$

Now, for $1 \le q < \infty$, we observe that for $p_j > q$, from Hölder's inequality with $\frac{p_j}{p_j-q}$ and $\frac{p_j}{q}$ we obtain

$$\begin{aligned}&\left(\iint_H J(x-y)|u_{p_j}(y)-u_{p_j}(x)|^q\,dx\,dy\right)^{1/q}\\ &\quad\le \left(\iint_H J(x-y)\,dy\,dx\right)^{\frac{p_j-q}{p_jq}}\left(\iint_H J(x-y)|u_{p_j}(y)-u_{p_j}(x)|^{p_j}\right)^{\frac{1}{p_j}}\\ &\quad\le \left(\iint_H J(x-y)\,dy\,dx\right)^{\frac{p_j-q}{p_jq}}(2p_jC)^{\frac{1}{p_j}}.\end{aligned}$$

Let $p_j\to\infty$ to obtain

$$\left(\iint_H J(x-y)|u_\infty(y)-u_\infty(x)|^q\,dx\,dy\right)^{1/q} \le \left(\iint_H J(x-y)\,dy\,dx\right)^{\frac{1}{q}}.$$

By taking $C(q) := \left(\iint_H J(x-y)\,dy\,dx\right)^{\frac{1}{q}}$ and letting $q \to \infty$ we obtain desired result. $\square$

We also have

$$\left(\iint_H J(x-y)|u_{p_j}(y)-u_{p_j}(x)|^q\,dx\,dy\right)^{1/q} \le C(p,q),$$

$$\begin{aligned}\left(\iint_H J(x-y)|u_\infty(y)-u_\infty(x)|^q\,dx\,dy\right)^{1/q}&\\ \le \liminf_{p_j\to\infty}&\left(\iint_H J(x-y)|u_{p_j}(y)-u_{p_j}(x)|^q\,dx\,dy\right)^{1/q}\\ \le \liminf_{p_j\to\infty}& C(p,q) \le 1.\end{aligned}$$

Now letting $q \to \infty$, we get

$$|u_\infty(y)-u_\infty(x)| \le 1 \quad \text{for a.e. } x,y\in H,\ x-y\in \operatorname{supp}(J).$$

Hence we conclude that

$$u_{p_j} \rightharpoonup u_\infty \text{ weakly in } L_q(\Gamma_r) \quad \text{and} \quad u_\infty \in B_\infty,$$

with

$$B_\infty := \big\{u : |u(x)-u(y)| \le 1 \text{ for a.e. } x,y\in H,\ x-y\in\operatorname{supp}(J)\big\}.$$

Lemma 3.6. *Let u_∞ be a limit of u_p (along a subsequence if necessary); then u_∞ is a solution to*

$$\sup_{u\in B_\infty}\int_{\Gamma_r} fu, \tag{3.2}$$

with

$$B_\infty = \big\{u : |u(x)-u(y)| \le 1 \textit{ for a.e. } x,y\in H,\ x-y\in\operatorname{supp}(J)\big\}.$$

Proof. Since u_p is the minimizer of J_p and $B_\infty \subset B_p$, we have, for all $v \in B_\infty$,

$$\begin{aligned}\frac{1}{2p}\iint_H J(x-y)|u_p(y)-u_p(x)|^p\,dx\,dy - \int_{\Gamma_r} fu_p&\\ \le \frac{1}{2p}\iint_H J(x-y)|v(y)-v(x)|^p\,dx\,dy - \int_{\Gamma_r} fv&\\ \le \frac{1}{2p}\iint_H J(x-y)\,dx\,dy - \int_{\Gamma_r} fv;&\end{aligned}$$

then,

$$-\int_{\Gamma_r} fu_p \le \frac{1}{2p}\iint_H J(x-y)\,dx\,dy - \int_{\Gamma_r} fv.$$

The idea now is to pass to the limit as p goes to infinity in the previous inequality. On the left-hand side, we can neglect the first integral, while

$$\int_{\Gamma_r} fu_p(x)\,dx \to \int_{\Gamma_r} fu_\infty(x)\,dx \quad \text{as } p\to\infty.$$

For the right-hand side, we know that the first integral goes to zero and then (3.2) holds. □

3.1. **Singular case.** We also include here the case in which the kernel J can be singular. For $0 < s < 1$,

$$\text{(P2)}\begin{cases} 0 = \displaystyle\int_{\Omega_r \cup \Gamma_r} \frac{1}{|x-y|^{n+ps}} |u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, & x \in \Omega_r; \\ f(x) = \displaystyle\int_{\Omega_r} \frac{1}{|x-y|^{n+ps}} |u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, & x \in \Gamma_r. \end{cases}$$

To deal with this problem, we consider the space

$$\mathbb{X}_{s,p} = \{u \in L_p(\Omega_r \cup \Gamma_r) : \|u\|_{s,p} < +\infty\},$$

where

$$\|u\|_{s,p} := \left(\|u\|^p_{L_p(\Gamma_r)} + \iint_H \frac{1}{|x-y|^{n+ps}} |u(y)-u(x)|^p \,dy\,dx \right)^{\frac{1}{p}}.$$

In the next lemma, as in Lemma 3.2, we will find a unique minimizer of $J_p(u)$ with singular kernel.

Lemma 3.7. *There is a unique solution u_p to the problem*

$$\min_{u \in B^s_p} J^s_p(u)$$

in the set

$$B^s_p := \left\{ u \in \mathbb{X}_{s,p} : \int_{\Gamma_r} u(x) = 0 \right\},$$

where

$$J^s_p(u) := \frac{1}{2p} \iint_H \frac{1}{|x-y|^{n+ps}} |u(y)-u(x)|^p \,dy\,dx - \int_{\Gamma_r} f u\,dx.$$

Proof. If we replace $J(x-y)$ by the singular kernel and use similar techniques as in the proof of Lemma 3.2, then we can obtain the desired result easily. □

Corollary 3.8. *Let u_p be the minimizer of $J^s_p(u)$ in B^s_p. As a consequence of Lemma 3.7, there exists a constant $K > 0$ independent of p such that $\|u_p\|_{s,p} \le K$. Then, by compact embedding (see [22]), we can extract a subsequence $\{u_{p_j}\} \subset u_p$ such that $u_{p_j} \to u_\infty$ in $L_q(\Gamma_r)$ for $p > q$.*

Lemma 3.9. *Let u_p be the minimizer of $J^s_p(u)$ in B^s_p. There exists a constant $C(p)$ such that*

$$\left(\iint_H \frac{1}{|x-y|^n} \frac{|u_p(y)-u_p(x)|^p}{|x-y|^{sp}} \,dx\,dy \right)^{1/p} \le C(p)$$

and

$$\lim_{p \to \infty} C(p) = C_\infty,$$

where $C_\infty = \max\{1, (\operatorname{diam}(\Omega))^\delta\}$ for small $\delta > 0$.

Proof. Let us take a fixed $v(x) \in C^\infty(\Omega)$ such that $\int_{\Gamma_r} v = 0$ and $\frac{v(x)-v(y)}{|x-y|^{s+\delta}} \leq 1$ for sufficiently small δ and for a.e. $x, y \in H$. Since u_p is the minimizer of J_p^s in B_p^s and $v \in B_p^s$, we have

$$\begin{aligned}
&\frac{1}{2p} \iint_H \frac{1}{|x-y|^{n+sp}} |u_p(y) - u_p(x)|^p \, dx \, dy \\
&\qquad \leq \int_{\Gamma_r} f u_p + \frac{1}{2p} \iint_H \frac{1}{|x-y|^{n+sp}} |v(y) - v(x)|^p \, dx \, dy - \int_{\Gamma_r} f v \\
&\qquad = \int_{\Gamma_r} f u_p - \int_{\Gamma_r} f v + \frac{1}{2p} \iint_H \frac{1}{|x-y|^{n-\delta p}} \Big[\frac{|v(y)-v(x)|}{|x-y|^{s+\delta}} \Big]^p \, dx \, dy.
\end{aligned}$$

For sufficiently large p, we have

$$\begin{aligned}
&\frac{1}{2p} \iint_H \frac{1}{|x-y|^{n+sp}} |u_p(y) - u_p(x)|^p \, dx \, dy \\
&\quad \leq \int_{\Gamma_r} f u_p - \int_{\Gamma_r} f v + \frac{1}{2p} |H| (\operatorname{diam}(\Omega))^{\delta p - n} \\
&\quad \leq c(\varepsilon) \|f\|_{L_{p'}(\Gamma_r)}^{p'} + \varepsilon \|u_p\|_{L_p(\Gamma_r)}^p - \int_{\Gamma_r} f v + \frac{1}{2p} |H| (\operatorname{diam}(\Omega))^{\delta p - n} \\
&\quad \leq c(\varepsilon) \|f\|_{L_{p'}(\Gamma_r)}^{p'} + \frac{\varepsilon}{c} \iint_H \frac{1}{|x-y|^{n+sp}} |u_p(y) - u_p(x)|^p \, dx \, dy \\
&\qquad + \int_{\Gamma_r} |f v| + \frac{1}{2p} |H| (\operatorname{diam}(\Omega))^{\delta p - n}.
\end{aligned}$$

This gives us

$$\begin{aligned}
&\Big(\frac{1}{2p} - \frac{\varepsilon}{c} \Big)^{\frac{1}{p}} \Big(\iint_H \frac{1}{|x-y|^n} \frac{|u_p(y) - u_p(x)|^p}{|x-y|^{sp}} \, dx \, dy \Big)^{1/p} \\
&\qquad \leq \Big(c(\varepsilon) \|f\|_{L_{p'}(\Gamma_r)}^{p'} + \int_{\Gamma_r} |f v| + \frac{1}{2p} |H| (\operatorname{diam}(\Omega))^{\delta p - n} \Big)^{\frac{1}{p}}.
\end{aligned}$$

Here

$$\Big(c(\varepsilon) \|f\|_{L_{p'}(\Gamma_r)}^{p'} + \int_{\Gamma_r} |f v| + \frac{1}{2p} |H| (\operatorname{diam}(\Omega))^{\delta p - n} \Big)^{\frac{1}{p}} \to C_\infty$$

and

$$\Big(\frac{1}{2p} - \frac{\varepsilon}{c} \Big)^{\frac{1}{p}} \to 1$$

as $p \to \infty$. Hence we obtain

$$\Big(\iint_H \frac{1}{|x-y|^n} \frac{|u_p(y) - u_p(x)|^p}{|x-y|^{sp}} \, dx \, dy \Big)^{1/p} \leq C(p) \quad \text{and} \quad \lim_{p \to \infty} C(p) = C_\infty,$$

where

$$C(p)=\frac{\left(c(\varepsilon)\|f\|^{p'}_{L_{p'}(\Gamma_r)}+\int_{\Gamma_r}|fv|+\frac{1}{2p}|H|(\operatorname{diam}(\Omega))^{\delta p-n}\right)^{\frac{1}{p}}}{\left(\frac{1}{2p}-\frac{\varepsilon}{c}\right)^{\frac{1}{p}}}. \qquad \square$$

Lemma 3.10. *Let u_p be the minimizer of $J^s_p(u)$ in B^s_p. There exists a constant $C(p,q)$ such that*

$$\lim_{p,q\to\infty} C(p,q)=C_\infty$$

and

$$\left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\left(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\right)^q dx\,dy\right)^{\frac{1}{q}}\le C(p,q).$$

Proof. We have

$$\begin{aligned}&\left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\left(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\right)^q dx\,dy\right)^{\frac{1}{q}}\\ &\qquad=\left(\iint_H (\operatorname{diam}(\Omega))^{\frac{nq}{p}-n}\frac{1}{(\operatorname{diam}(\Omega))^{\frac{nq}{p}}}\left(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\right)^q dx\,dy\right)^{\frac{1}{q}}.\end{aligned}$$

Now, for $1\le q<\infty$, we observe that for $p>q$, from Hölder's inequality we obtain

$$\begin{aligned}&\left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\left(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\right)^q dx\,dy\right)^{\frac{1}{q}}\\ &\quad\le\left(\iint_H \left((\operatorname{diam}(\Omega))^{\frac{nq}{p}-n}\right)^{\frac{p}{p-q}}dx\,dy\right)^{\frac{p-q}{pq}}\\ &\qquad\times\left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\left(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\right)^p dx\,dy\right)^{\frac{1}{p}}\\ &\quad\le\left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\,dx\,dy\right)^{\frac{p-q}{pq}}\\ &\qquad\times\left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\left(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\right)^p dx\,dy\right)^{\frac{1}{p}}\\ &\quad\le\left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\,dx\,dy\right)^{\frac{p-q}{pq}}\left(\iint_H \frac{1}{|x-y|^n}\frac{|u_p(y)-u_p(x)|^p}{|x-y|^{sp}}\,dx\,dy\right)^{1/p}\\ &\quad\le\left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\,dx\,dy\right)^{\frac{p-q}{pq}}C(p).\end{aligned}$$

Hence we obtain

$$\left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\left(\frac{|u_p(y)-u_p(x)|}{|x-y|^s}\right)^q dx\,dy\right)^{\frac{1}{q}} \le C(p,q),$$

where

$$C(p,q)=\left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\,dx\,dy\right)^{\frac{p-q}{pq}} C(p) \quad \text{and} \quad \lim_{p,q\to\infty} C(p,q)=C_\infty. \qquad \square$$

Therefore, as a consequence of Lemma 3.10, we have

$$\begin{aligned}
&\left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\left(\frac{|u_\infty(y)-u_\infty(x)|}{|x-y|^s}\right)^q dx\,dy\right)^{1/q}\\
&\qquad\le \liminf_{p_j\to\infty}\left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\left(\frac{|u_{p_j}(y)-u_{p_j}(x)|}{|x-y|^s}\right)^q dx\,dy\right)^{1/q}\\
&\qquad\le \liminf_{p_j\to\infty} C(p_j,q) \le \left(\iint_H \frac{1}{(\operatorname{diam}(\Omega))^n}\,dx\,dy\right)^{\frac{1}{q}} C_\infty.
\end{aligned}$$

Now letting $q\to\infty$, we get

$$|u_\infty(y)-u_\infty(x)| \le \max\{1,(\operatorname{diam}(\Omega))^\delta\}|x-y|^s \quad \text{for a.e. } x,y\in H,\ x\ne y.$$

Then we have

$$u_{p_j}\to u_\infty \text{ in } L_q(\Gamma_r) \quad \text{and} \quad u_\infty\in B^s_\infty,$$

with

$$B^s_\infty := \left\{u : \tfrac{|u(y)-u(x)|}{|x-y|^s} \le \max\{1,(\operatorname{diam}(\Omega))^\delta\} \text{ a.e. } x,y\in H \text{ such that } x\ne y\right\}.$$

Now our aim is to obtain an equation satisfied by the limit u_∞ in the usual viscosity sense.

Theorem 3.11. *Assume that $f\in C(\hat{\Omega})$. Then u_∞ is the solution of the problem*

$$\begin{cases}
\sup\limits_{x\ne y}\dfrac{u_\infty(y)-u_\infty(x)}{|y-x|^s}+\inf\limits_{x\ne y}\dfrac{u_\infty(y)-u_\infty(x)}{|y-x|^s}=0, & x\in\Omega_r;\\
\sup\limits_{x\ne y}\dfrac{u_\infty(y)-u_\infty(x)}{|x-y|^s}-\max\left\{\sup\limits_{x\ne y}\dfrac{-u_\infty(y)+u_\infty(x)}{|x-y|^s},1\right\}=0, & f(x)>0,\ x\in\Gamma_r;\\
\max\left\{\sup\limits_{x\ne y}\dfrac{u_\infty(y)-u_\infty(x)}{|x-y|^s},1\right\}-\sup\limits_{x\ne y}\dfrac{-u_\infty(y)+u_\infty(x)}{|x-y|^s}=0, & f(x)<0,\ x\in\Gamma_r
\end{cases} \tag{3.3}$$

in the usual viscosity sense.

Proof. Let us start by showing that u_∞ is a viscosity subsolution of (3.3). Let $x_0\in\Omega_r$ and let φ be a $C^2(\hat{\Omega})$ function such that $\varphi-u_\infty$ has a minimum at x_0 and $u(x_0)=\varphi(x_0)$. We want to prove that

$$\sup_{x_0\ne y}\frac{\varphi(y)-\varphi(x_0)}{|y-x_0|^s}+\inf_{x_0\ne y}\frac{\varphi(y)-\varphi(x_0)}{|y-x_0|^s}\ge 0.$$

Thanks to the uniform convergence of u_p to u_∞ (see [15, Proposition 6.1]), there exists a sequence of points $x_p \to x_0$ such that $\varphi - u_p$ has a minimum at $x_p \in \Omega_r$; then for all $y \in \Omega_r \cup \Gamma_r$ we have

$$\varphi(y) - u_p(y) \geq \varphi(x_p) - u_p(x_p) \Rightarrow \varphi(y) - \varphi(x_p) \geq u_p(y) - u_p(x_p).$$

Therefore,

$$|\varphi(y) - \varphi(x_p)|^{p-2}(\varphi(y) - \varphi(x_p)) \geq |u_p(y) - u_p(x_p)|^{p-2}(u_p(y) - u_p(x_p))$$

and

$$\begin{aligned}\frac{1}{|x_p - y|^{n+ps}}&|\varphi(y) - \varphi(x_p)|^{p-2}(\varphi(y) - \varphi(x_p)) \\ &\geq \frac{1}{|x_p - y|^{n+ps}}|u_p(y) - u_p(x_p)|^{p-2}(u_p(y) - u_p(x_p)).\end{aligned}$$

Integrating on $\Omega_r \cup \Gamma_r$,

$$\begin{aligned}\int_{\Omega_r\cup\Gamma_r} &\frac{1}{|x_p - y|^{n+ps}}|\varphi(y) - \varphi(x_p)|^{p-2}(\varphi(y) - \varphi(x_p))\, dy \\ &\geq \underbrace{\int_{\Omega_r\cup\Gamma_r} \frac{1}{|x_p - y|^{n+ps}}|u_p(y) - u_p(x_p)|^{p-2}(u_p(y) - u_p(x_p))\, dy.}_{=0}\end{aligned}$$

Then, we have

$$\begin{aligned}\int_{\Omega_r\cup\Gamma_r\cap\{\varphi(y)<\varphi(x_p)\}} &\frac{1}{|x_p - y|^{n+ps}}|\varphi(y) - \varphi(x_p)|^{p-1}\, dy \\ &\leq \int_{\Omega_r\cup\Gamma_r\cap\{\varphi(y)>\varphi(x_p)\}} \frac{1}{|x_p - y|^{n+ps}}|\varphi(y) - \varphi(x_p)|^{p-1}\, dy;\end{aligned}$$

therefore we get

$$\begin{aligned}&\left(\int_{\Omega_r\cup\Gamma_r\cap\{\varphi(y)<\phi(x_p)\}} \frac{1}{|x_p - y|^{n+ps}}|\varphi(y) - \phi(x_p)|^{p-1}\, dy\right)^{1/(p-1)} \\ &\qquad\leq \left(\int_{\Omega_r\cup\Gamma_r\cap\{\varphi(y)>\varphi(x_p)\}} \frac{1}{|x_p - y|^{n+ps}}|\varphi(y) - \varphi(x_p)|^{p-1}\, dy\right)^{1/(p-1)},\end{aligned}$$

and taking $p \to \infty$ we conclude that

$$\sup_{\varphi(y)>\varphi(x_0)} \frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^s} \geq \sup_{\varphi(y)<\varphi(x_0)} \frac{-\varphi(y) + \varphi(x_0)}{|y - x_0|^s}.$$

Hence,

$$\sup_{x_0\neq y} \frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^s} + \inf_{x_0\neq y} \frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^s} \geq 0.$$

Now, for the second equation, let us consider $x_0 \in \Gamma_r$ and let φ be a $C^2(\hat{\Omega})$ function such that $\varphi - u_\infty$ has a minimum at x_0 and $u_\infty(x_0) = \varphi(x_0)$.

Thanks to the uniform convergence of u_p to u_∞, there exists a sequence of points $x_p \to x_0$ such that $\varphi - u_p$ has a minimum at $x_p \in \Gamma_r$; then for all $y \in \Omega_r$ we have

$$\begin{aligned}\int_{\Omega_r} \frac{1}{|x_p - y|^{n+ps}} |\varphi(y) - \varphi(x_p)|^{p-2}(\varphi(y) - \varphi(x_p))\, dy \\ \geq \int_{\Omega_r} \frac{1}{|x_p - y|^{n+ps}} |u_p(y) - u_p(x_p)|^{p-2}(u_p(y) - u_p(x_p))\, dy = f(x_p).\end{aligned}$$

Then, we have

$$\begin{aligned}\int_{\Omega_r \cap \{\varphi(y) > \varphi(x_p)\}} \frac{1}{|x_p - y|^{n+ps}} |\varphi(y) - \varphi(x_p)|^{p-1}\, dy \\ \geq \int_{\Omega_r \cap \{\varphi(y) < \varphi(x_p)\}} \frac{1}{|x_p - y|^{n+ps}} |\varphi(y) - \varphi(x_p)|^{p-1}\, dy + f(x_p),\end{aligned}$$

and therefore we get

$$\begin{aligned}\left(\int_{\Omega_r \cap \{\varphi(y) > \varphi(x_p)\}} \frac{1}{|x_p - y|^{n+ps}} |\varphi(y) - \varphi(x_p)|^{p-1}\, dy\right)^{1/(p-1)} \\ \geq \left(\int_{\Omega_r \cap \{\varphi(y) < \varphi(x_p)\}} \frac{1}{|x_p - y|^{n+ps}} |\varphi(y) - \varphi(x_p)|^{p-1}\, dy + f(x_p)\right)^{1/(p-1)}.\end{aligned}$$

Recall that given two sequences of nonnegative real numbers $a_p, b_p \subset \mathbb{R}$, where $a_p^{\frac{1}{p}} \to a$ and $b_p^{\frac{1}{p}} \to b$, we have $(a_p + b_p)^{\frac{1}{p-1}} \to \max\{a, b\}$.

In the case $f(x_p) > 0$, if we take $p \to \infty$ and use the above property we conclude that

$$\sup_{x_0 \neq y} \frac{\varphi(y) - \varphi(x_0)}{|x_0 - y|^s} \geq \max\left\{\sup_{x_0 \neq y} \frac{-\varphi(y) + \varphi(x_0)}{|x_0 - y|^s}, 1\right\}.$$

Observe also that, in the case $f(x_p) < 0$,

$$\begin{aligned}\left(\int_{\Omega_r \cap \{\varphi(y) > \varphi(x_p)\}} \frac{1}{|x_p - y|^{n+ps}} |\varphi(y) - \varphi(x_p)|^{p-1} dy - f(x_p)\right)^{1/(p-1)} \\ \geq \left(\int_{\Omega_r \cap \{\varphi(y) < \varphi(x_p)\}} \frac{1}{|x_p - y|^{n+ps}} |\varphi(y) - \varphi(x_p)|^{p-1} dy\right)^{1/(p-1)},\end{aligned}$$

and taking $p \to \infty$, we obtain

$$\max\left\{\sup_{x_0 \neq y} \frac{\varphi(y) - \varphi(x_0)}{|x_0 - y|^s}, 1\right\} \geq \sup_{x_0 \neq y} \frac{-\varphi(y) + \varphi(x_0)}{|x_0 - y|^s}.$$

We conclude that u_∞ is a viscosity subsolution to

$$\begin{cases} \sup_{x \neq y} \frac{u_\infty(y) - u_\infty(x)}{|x-y|^s} - \max\left\{\sup_{x \neq y} \frac{-u_\infty(y) + u_\infty(x)}{|x-y|^s}, 1\right\} = 0, & f(x) > 0, \\ \max\left\{\sup_{x \neq y} \frac{u_\infty(y) - u_\infty(x)}{|x-y|^s}, 1\right\} - \sup_{x \neq y} \frac{-u_\infty(y) + u_\infty(x)}{|x-y|^s} = 0, & f(x) < 0 \end{cases}$$

for $x \in \Gamma_r$.

In order to prove that u is also a supersolution, let $x_0 \in \Omega_r$ and let ψ be a $C^2(\hat{\Omega})$ function such that $\psi - u_\infty$ has a maximum at x_0 and $u_\infty(x_0) = \psi(x_0)$.

As in the previous case, we can infer from the uniform convergence of $u_p \to u_\infty$ that there exists a sequence of points $x_p \to x_0$ such that $\psi - u_p$ has a maximum at x_p, with $u_p(x_p) = \psi(x_p)$. By using similar arguments, we have

$$\sup_{\psi(y)>\psi(x_0)} \frac{\psi(y)-\psi(x_0)}{|y-x_0|^s} + \inf_{\psi(y)<\psi(x_0)} \frac{\psi(y)-\psi(x_0)}{|y-x_0|^s} \leq 0.$$

Let $x_0 \in \Gamma_r$ and ψ be a $C^2(\hat{\Omega})$ function such that $\psi - u_\infty$ has a maximum at x_0 and $u_\infty(x_0) = \psi(x_0)$. Thanks to the uniform convergence of u_p to u_∞, there exists a sequence of points $x_p \to x_0$ such that $\psi - u_p$ has a maximum at $x_p \in \Gamma_r$; then for all $y \in \Omega_r$ we have

$$\sup_{x_0 \neq y} \frac{\psi(y)-\psi(x_0)}{|x_0-y|^s} \leq \max\Big\{ \sup_{x_0\neq y} \frac{-\psi(y)+\psi(x_0)}{|x_0-y|^s}, 1\Big\}, \quad f(x_0) > 0;$$

$$\max\Big\{ \sup_{x_0\neq y} \frac{\psi(y)-\psi(x_0)}{|x_0-y|^s}, 1\Big\} \leq \sup_{x_0\neq y} \frac{-\psi(y)+\psi(x_0)}{|x_0-y|^s}, \quad f(x_0) < 0.$$

We conclude that u_∞ is a viscosity solution to (3.3). ☐

4. Parabolic case

Recall from the Introduction that the nonlocal p-Laplacian evolution problem (P1):

$$\begin{cases} 0 = \displaystyle\int_{\Omega_r \cup \Gamma_r} J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy, & x \in \Omega_r,\ t>0; \\ \dfrac{\partial u}{\partial t}(x,t) = \displaystyle\int_{\Omega_r} J(x-y)|u(y,t)-u(x,t)|^{p-2} & \\ \qquad\qquad\qquad \times(u(y,t)-u(x,t))dy + f(x,t), & x \in \Gamma_r,\ t>0; \\ u(x,0) = u_0(x), & x \in \Gamma_r. \end{cases}$$

Our aim in this section concerns the limit as $p \to \infty$ in (P1). Firstly, let us show the existence and uniqueness of solutions to the problem (P1) by using abstract semigroup theory. Let us define the functional $E_p(u)$ associated with (P1):

$$E_p(u) := \begin{cases} \dfrac{1}{2p} \displaystyle\iint_H J(x-y)|u(y,t)-u(x,t)|^p\,dy\,dx & \text{if } u \in A_p, \\ +\infty & \text{if } u \notin A_p, \end{cases}$$

where

$$A_p := \Big\{ u \in L_p(H) : \iint_H J(x-y)|u(y,t)-u(x,t)|^p\,dy\,dx < +\infty \Big\}.$$

Then the nonlocal problem can be written as the abstract Cauchy problem associated to the subdifferential of E_p, that is:

$$(\text{P1-s})\begin{cases} f(\cdot,t)-\dfrac{\partial u}{\partial t}(\cdot,t)=\partial E_p(u(\cdot,t)), & \text{a.e. } t\in(0,T);\\ u(x,0)=u_0(x), & x\in\Gamma_r. \end{cases}$$

With a formal computation, taking limits, we arrive at the functional

$$E_\infty(u)=\begin{cases}0 & \text{if } u\in A_\infty,\\ +\infty & \text{if } u\notin A_\infty,\end{cases}$$

where

$$A_\infty:=\{u:|u(x)-u(y)|\le 1 \text{ for a.e. } x,y\in H,\ |x-y|\in\operatorname{supp}(J)\}.$$

Then the nonlocal limit problem can be written as

$$(\text{P}_\infty)\begin{cases} f(\cdot,t)-\dfrac{\partial u}{\partial t}(\cdot,t)\in\partial E_\infty(u(\cdot,t)), & \text{a.e. } t\in(0,T),\\ u(x,0)=u_0(x), & x\in\Gamma_r. \end{cases}$$

We have the following existence result.

Theorem 4.1. *Suppose $p>1$ and let $u_0\in L_p(\Gamma_r)$. Then, for any $T>0$ and $f(t,x)\in C([0,T]\times\Gamma_r)$, there exists a unique solution $u_p(x,t)\in C([0,\infty);L_p(\Gamma_r))$ to* (P1).

For the proof of this theorem, as we have mentioned, we will use the perspective of nonlinear semigroup theory. To proceed with this task, we first need the following lemma, where we prove that the operator $\partial E_p(u)$ satisfies adequate conditions to apply nonlinear semigroup theory to solve (P1).

Lemma 4.2. *The operator $\partial E_p(u)=B_p(u)$ is m-accretive in $L_p(\Gamma_r)$.*

Proof. For the proof of m-accretiveness we should have the following two conditions:

(i) Given $u_1,u_2\in\operatorname{Dom}(B_p)$ and $q\in P_0$,

$$\int_{\Gamma_r}(B_p(u_1)(x)-B_p(u_2)(x))q(u_1(x)-u_2(x))\,dx\ge 0,$$

where

$$P_0=\{q\in C^\infty(\mathbb{R}):0\le q'\le 1,\ \operatorname{supp}(q') \text{ is compact and } 0\notin\operatorname{supp}(q)\}.$$

(ii) B_p satisfies the range condition

$$L_p(\Gamma_r)\subset R(B_p+I).$$

By [9, Theorem 3.4], (i) and (ii) are proved. □

Proof of Theorem 4.1. By [9, Corollary A.3], Theorem 4.1 is proved. □

Theorem 4.3. *Let $T > 0$, $f(t,x) \in C([0,T] \times \Gamma_r)$, $u_p(x,t)$ be the solution to* (P1) *with a fixed initial condition*

$$u_0 \in \overline{A_\infty}^{L^2(\Gamma_r)}.$$

Then if u_∞ is the unique solution of P_∞*,*

$$u_p \to u_\infty$$

as $p \to \infty$ in $C([0,T] : L^2(\Gamma_r))$, that is,

$$\lim_{p\to\infty} \sup_{t\in[0,T]} \|u_p(\cdot,t) - u_\infty(\cdot,t)\|_{L^2(\Gamma_r)} = 0.$$

Proof. Let us show that the functional

$$E_p(u) = \frac{1}{2p} \iint_H J(x-y)|u(y,t) - u(x,t)|^p \, dy \, dx$$

converges to

$$E_\infty(u) = \begin{cases} 0 & \text{if } u \in A_\infty, \\ +\infty & \text{if } u \notin A_\infty \end{cases}$$

as $p \to \infty$ in the sense of Mosco. For the Mosco convergence of E_p to E_∞, first let us prove that

$$\mathrm{Epi}(E_\infty) \subset s\text{-}\liminf_{p\to\infty} \mathrm{Epi}(E_p).$$

Let us take $(u,k) \in \mathrm{Epi}(E_\infty)$; then our claim is that there exists (u_p, k_p) such that $u_p \to u$ in $L_2(H)$ and $k_p \to k$. Now take

$$u_p = u \quad \text{and} \quad k_p = c_p + k, \text{ where } c_p = E_p(u).$$

Here, if $u \in A_\infty$ then $E_\infty(u) = 0$ and $k \geq 0$. Also, if $c_p \to 0$ then

$$k_p = c_p + k \to k \quad \text{and} \quad k_p \geq E_p(u).$$

This gives us $(u_p, k_p) \in \mathrm{Epi}(E_p)$ and since $(u_p, k_p) \to (u,k)$ we obtain $(u,k) \in s\text{-}\liminf_{p\to\infty} \mathrm{Epi}(E_p)$.

On the other hand, if $u \notin A_\infty$ then $E_\infty(u) = \infty$ and $(u, +\infty) \in \mathrm{Epi}(E_\infty)$. By taking $u_p = u$, $k_p = +\infty$, we get that $k_p = +\infty \to +\infty$, $u_p \to u$ in $L_2(H)$ and $(u_p, +\infty) \in \mathrm{Epi}(E_p)$, so we obtain $(u,k) \in s\text{-}\liminf_{p\to\infty} \mathrm{Epi}(E_p)$. Secondly, let us prove that

$$w\text{-}\limsup_{p\to\infty} \mathrm{Epi}(E_p) \subset \mathrm{Epi}(E_\infty).$$

Given $(u,k) \in w\text{-}\limsup_{p\to\infty} \mathrm{Epi}(E_p)$, there exists a sequence $(u_p, k_p) \in \mathrm{Epi}(E_p)$ such that

$$u_p \rightharpoonup u \text{ in } L_2(H), \quad k_p \to k \text{ in } \mathbb{R}.$$

We can assume that there exists a constant $\tilde{c}$ such that $k_p \leq \tilde{c}$. Otherwise $k_p \to \infty$ and since $k = +\infty \geq E_\infty(u)$ we get $(u, +\infty) \in \mathrm{Epi}(E_\infty)$.

On the other hand, we have

$$\frac{1}{2p} \iint_H J(x-y)|u_p(y,t) - u_p(x,t)|^p \, dy \, dx \leq \tilde{c}$$

and

$$\left(\iint_H J(x-y)|u_p(y,t)-u_p(x,t)|^p\,dy\,dx\right)^{1/p} \leq (2p\tilde{c})^{1/p}.$$

We have also

$$\left(\iint_H J(x-y)|u_p(y)-u_p(x)|^q\,dx\,dy\right)^{1/q}$$
$$= \left(\iint_H (J(x-y))^{1-\frac{q}{p}}\left[(J(x-y))^{\frac{q}{p}}|u_p(y)-u_p(x)|^q\right]\right)^{\frac{1}{q}}.$$

Now, for $1 \leq q < \infty$, we observe that for $p > q$, from Hölder's inequality with $\frac{p}{p-q}$ and $\frac{p}{q}$ we obtain

$$\begin{aligned}&\left(\iint_H J(x-y)|u_p(y)-u_p(x)|^q\,dx\,dy\right)^{1/q}\\ &\quad\leq \left(\iint_H J(x-y)\,dy\,dx\right)^{\frac{p-q}{pq}}\left(\iint_H J(x-y)|u_p(y)-u_p(x)|^p\right)^{\frac{1}{p}}\\ &\quad\leq \left(\iint_H J(x-y)\,dy\,dx\right)^{\frac{p-q}{pq}}(2p\tilde{c})^{1/p}.\end{aligned}$$

Let $p \to \infty$ to obtain

$$\left(\iint_H J(x-y)|u(y)-u(x)|^q\,dx\,dy\right)^{1/q} \leq \left(\iint_H J(x-y)\,dy\,dx\right)^{\frac{1}{q}}.$$

Now letting $q \to \infty$, we get

$$|u(y)-u(x)| \leq 1 \quad \text{for a.e. } x,y \in H,\ x-y \in \operatorname{supp}(J).$$

Hence we obtain $u \in A_\infty$; this means that $E_\infty(u) = 0$ and $k \geq 0$, and so we get that $(u,k) \in \operatorname{Epi}(E_\infty)$. □

5. Explicit solutions

In this section, we give some explicit examples of limit solutions for certain problems. Firstly we will give an explicit example in the elliptic case:

$$\text{(P1s)}\begin{cases} 0 = \displaystyle\int_{\Omega_r\cup\Gamma_r} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, & x \in \Omega_r; \\ f(x) = \displaystyle\int_{\Omega_r} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))dy, & x \in \Gamma_r. \end{cases}$$

In order to verify that a function $u_\infty(x)$ is a limit solution to (P1s) we need to check that $u_\infty(x)$ is the solution of

$$\sup_{u\in B_\infty}\int_{\Gamma_r} fu,$$

where

$$B_\infty = \big\{u : |u(x) - u(y)| \le 1 \text{ for a.e. } x, y \in H,\ x - y \in \operatorname{supp}(J) = B_R(0)\big\}.$$

Example 5.1. We consider the 1-dimensional case with $\Omega_r = (0,1)$, $\Gamma_r = (-\delta, 0) \cup (1, 1+\delta)$,

$$f(x) = \begin{cases} 1, & x \in (1, 1+\delta), \\ -1, & x \in (-\delta, 0), \end{cases}$$

and

$$u_\infty(x) = \begin{cases} c, & x \in (1, 1+\delta), \\ c-1, & x \in (1-\delta, 1), \\ c-2, & x \in (1-2\delta, 1-\delta), \\ \quad\vdots & \quad\quad\vdots \\ c-(k-1), & x \in (1-(k-1)\delta, 1-(k-2)\delta), \\ c-k, & x \in (-\delta, 0). \end{cases}$$

Indeed, if we take $k = \frac{1}{\delta} + 1$ then we obtain the interval $(-\delta, 0)$. On the other hand, $\int_{\Gamma_r} u_\infty(x)\, dx = 0$, then we have

$$\int_{\Gamma_r} u_\infty(x)\, dx = c\delta + (c-k)\delta = c\delta + (c - (\frac{1}{\delta} + 1))\delta = 0,$$

and hence

$$c = \frac{1}{2} + \frac{1}{2\delta}.$$

From our choice of u_∞, clearly $|u_\infty(x) - u_\infty(y)| \le 1$ for $|x - y| \le \delta$ and also u_∞ is the solution of $\sup_{u \in B_\infty} \int_{\Gamma_r} fu$.

Now we will give some explicit examples for the parabolic case:

$$(\mathrm{P}_\infty) \begin{cases} f(\cdot, t) - \dfrac{\partial u}{\partial t}(\cdot, t) \in \partial E_\infty(u(\cdot, t)) & \text{a.e. } t \in (0, T), \\ u(x, 0) = u_0(x), \end{cases}$$

where

$$E_\infty(u) = \begin{cases} 0 & \text{if } u \in A_\infty, \\ +\infty & \text{if } u \notin A_\infty. \end{cases}$$

In order to verify that a function $u(x,t)$ is a solution to (P_∞) we need to check that

$$\int_{\Gamma_r} \Big(f - \frac{\partial u}{\partial t}\Big)(v - u) \le 0 \quad \text{for all } v \in D(E_\infty). \tag{5.1}$$

Example 5.2. We consider the 1-dimensional case with $\Omega_r = (0,1)$, $\Gamma_r = (-\delta, 0) \cup (1, 1+\delta)$, and take

$$f(x,t) := \begin{cases} 1, & x \in (1, 1+\delta),\ t \ge 0, \\ 0, & x \in (-\delta, 0),\ t \ge 0, \end{cases}$$

and as initial data

$$u_0(x) = 0.$$

Now, let us find the solution by looking at its evolution between some critical times. Let us take $u_\infty(x,t)$ as follows:

$$u_\infty(x,t) = \begin{cases} t, & x \in [1, 1+\delta),\, t \in [0, t_1), \\ 0, & \text{otherwise}. \end{cases}$$

Clearly from our choice of $u_{\infty(x,t)}$, if $t_1 \leq 1$ and $|x-y| \leq \delta$ then $|u_\infty(x,t) - u_\infty(y,t)| \leq 1$, so we get that $u_\infty(x,t) \in A_\infty$ and (5.1) holds. Hence, for small times $t_1 \leq 1$, the solution to (P_∞) is given by $u_\infty(x,t)$.

For times greater than t_1, let us take $u_\infty(x,t)$ as follows:

$$u_\infty(x,t) = \begin{cases} t, & x \in [1, 1+\delta),\, t \in [t_1, t_2), \\ t-1, & x \in [1-\delta, 1),\, t \in [t_1, t_2), \\ 0, & \text{otherwise}. \end{cases}$$

If $t_2 \leq 2$, $\delta \leq 1$, then $|u_\infty(x,t) - u_\infty(y,t)| \leq 1$ as $|x-y| \leq \delta$, so we get that $u_\infty(x,t) \in A_\infty$ and (5.1) holds.

Now, it is easy to generalize and verify the following general formula that describes the solution for time $t_j \leq j$ for any given integer j. We have

$$u_\infty(x,t) = \begin{cases} t, & x \in x \in [1, 1+\delta),\, t \in [t_{j-1}, t_j), \\ t-1, & x \in [1-\delta, 1),\, t \in [t_{j-1}, t_j), \\ t-2, & x \in [1-2\delta, 1-\delta),\, t \in [t_{j-1}, t_j), \\ \quad\vdots & \quad\vdots \\ t-(j-1), & x \in [1-(j-1)\delta, 1-(j-2)\delta),\, t \in [t_{j-1}, t_j), \\ 0, & \text{otherwise}, \end{cases}$$

where $\delta \leq \frac{1}{j-1}$ for $j \geq 2$.

Example 5.3. We consider the 1-dimensional case with $\Omega_r = (0,1)$, $\Gamma_r = (-\delta, 0) \cup (1, 1+\delta)$, and take

$$f(x,t) := \begin{cases} 1, & x \in (1, 1+\delta),\, t \geq 0, \\ -1, & x \in (-\delta, 0),\, t \geq 0, \end{cases}$$

and as initial data

$$u_0(x) = 0.$$

As in the previous example, the evolution follows the same scheme differently from $(-\delta, 0)$. Let us take $u_\infty(x,t)$ as follows:

$$u_\infty(x,t) = \begin{cases} t, & x \in [1, 1+\delta),\, t \in [0, t_1), \\ -t, & x \in (-\delta, 0),\, t \in [0, t_1), \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, if $t_1 \leq 1$ and $|x-y| \leq \delta$ then $|u_\infty(x,t) - u_\infty(y,t)| \leq 1$, so we get that $u_\infty(x,t) \in A_\infty$ and (5.1) holds. Hence, for small times $t_1 \leq 1$, the solution to (P_∞) is given by $u_\infty(x,t)$.

For times greater than t_1, let us take $u_\infty(x,t)$ as follows:

$$u_\infty(x,t) = \begin{cases} t, & x \in [1, 1+\delta),\ t \in [t_1, t_2), \\ t-1, & x \in [1-\delta, 1),\ t \in [t_1, t_2), \\ -(t-1), & x \in (-\delta, 0),\ t \in [t_1, t_2), \\ 0, & \text{otherwise.} \end{cases}$$

If $t_2 \leq 2$, $0 < \delta < 1$, then $|u_\infty(x,t) - u_\infty(y,t)| \leq 1$ as $|x-y| \leq \delta$, so we get that $u_\infty(x,t) \in A_\infty$ and (5.1) holds.

Now, let us generalize and verify the following general formula that describes the solution for time $t_j \leq j$ for any given integer j. We have

$$u_\infty(x,t) = \begin{cases} t, & x \in [1, 1+\delta),\ t \in [t_{j-1}, t_j), \\ t-1, & x \in [1-\delta, 1),\ t \in [t_{j-1}, t_j), \\ t-2, & x \in [1-2\delta, 1-\delta),\ t \in [t_{j-1}, t_j), \\ \quad\vdots & \qquad\vdots \\ t-(j-1), & x \in [1-(j-1)\delta, 1-(j-2)\delta),\ t \in [t_{j-1}, t_j), \\ 0, & x \in (0, 1-(j-1)\delta),\ t \in [t_{j-1}, t_j), \\ -(t-(j-1)), & x \in (-\delta, 0),\ t \in [t_{j-1}, t_j), \\ 0, & \text{otherwise,} \end{cases}$$

where $\delta < \frac{1}{j-1}$.

Example 5.4. For two or more dimensions we can obtain similar examples. Consider a bounded domain $\Omega_r := B_1(0) \subset \mathbb{R}^N$ and $\Gamma_r := B_{1+\delta}(0) \setminus B_1(0)$, where

$$B_1(0) = \left\{x \in \mathbb{R}^N : \|x\| < 1\right\} \quad \text{and} \quad B_{1+\delta}(0) = \left\{x \in \mathbb{R}^N : \|x\| < 1+\delta\right\},$$

and take

$$f(x,t) = 1 \quad \text{for all } x \in \Gamma_r \text{ and all } t \geq 0$$

and as initial data

$$u_0(x) = 0.$$

Now, we can generalize and verify the following general formula that describes the solution for time $t_j \leq j$ for any given integer j. We have

$$u_\infty(x,t) = \begin{cases} t, & x \in B_{1+\delta}(0) \setminus B_1(0),\ t \in [t_{j-1}, t_j), \\ t-1, & x \in B_1(0) \setminus B_{1-\delta}(0),\ t \in [t_{j-1}, t_j), \\ t-2, & x \in B_{1-\delta}(0) \setminus B_{1-2\delta}(0),\ t \in [t_{j-1}, t_j), \\ \quad\vdots & \qquad\vdots \\ t-(j-1), & x \in B_{1-(j-2)\delta}(0) \setminus B_{1-(j-1)\delta}(0),\ t \in [t_{j-1}, t_j), \\ 0, & \text{otherwise,} \end{cases}$$

where $\delta < \frac{1}{j-1}$. It is clear that $u_\infty(x,t) \in A_\infty$ and (5.1) holds.

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