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ON MAPS PRESERVING THE JORDAN PRODUCT OF C-SYMMETRIC OPERATORS

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Abstract. Given a conjugation C on a complex separable Hilbert space H, a bounded linear operator A acting on H is said to be C-symmetric if $A = CA^*C$. In this paper, we provide a complete description to all those maps on the algebra of linear operators acting on a finite dimensional Hilbert space that preserve the Jordan product of C-symmetric operators, in both directions, for every conjugation C on H.

1. Introduction

Throughout this paper H will denote a finite dimensional complex Hilbert space of dimension at least three with the inner product $\langle .,. \rangle$. The algebra of all linear operators acting on H is denoted by $\mathcal{B}(H)$. A *conjugation* on H is an anti-linear operator C on H that satisfies these conditions:

(i) C is isometric: $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in H$;
(ii) C is involutive: $C^2 = I$.

An operator $A \in \mathcal{B}(H)$ is then said to be C-*symmetric*, or simply *complex symmetric* [6], if $A = CA^*C$, where A^* denotes the adjoint of A. Clearly, C-symmetric operators form a $*$-closed subspace of $\mathcal{B}(H)$.

It is well known that, for every conjugation C on H, we can find an orthonormal basis $\{e_i\}$ of H such that $Ce_i = e_i$ for every $i \geq 1$. Moreover, for every $A \in \mathcal{B}(H)$, we have

$$A \text{ is } C\text{-symmetric} \quad \Leftrightarrow \quad \langle Ae_i, e_j \rangle = \langle Ae_j, e_i \rangle \quad \text{for all } i, j \geq 1. \tag{1.1}$$

In other words, A is complex symmetric if and only if it has a symmetric (i.e., self-transpose) matrix representation with respect to some orthonormal basis, see [6]. The reader is referred to [5, 10, 9, 12, 13] and the references therein for more details about complex symmetry and its connection to other subjects.

The so-called *non-linear preserving problem* consists in characterizing those maps Φ on matrix algebras, linear operators algebras or, more generally, Banach

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algebras, that preserve certain properties or subsets under a given operation such as the usual product, the Jordan product or Lie product without assuming any linearity or additivity on Φ. It was shown in [4] that a one-to-one map Φ, on the real space of hermitian operators in $\mathcal{B}(H)$, preserves Jordan zero product in both directions if and only if it has one of the following forms:

$$A \mapsto f(A)UAU^* \quad \text{or} \quad A \mapsto f(A)UCACU^*,$$

where f is a non-vanishing real-valued function, $U \in \mathcal{B}(H)$ is a unitary operator and C is a conjugation on H. Non-linear preserver problems have received much attention over the last years; see, for instance, [3, 11, 14].

The Jordan product of two operators $A, B \in \mathcal{B}(H)$ is the operation given by $A \circ B = AB + BA$. For $\Lambda \subset \mathcal{B}(H)$, we say that a map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ *preserves the Jordan product of* Λ *in both directions* if, for all $A, B \in \mathcal{B}(H)$,

$$A \circ B \in \Lambda \quad \Leftrightarrow \quad \Phi(A) \circ \Phi(B) \in \Lambda.$$

The purpose of this paper is to describe all those maps on $\mathcal{B}(H)$ that preserve the Jordan product of the class of C-symmetric operators, in both directions, for every conjugation C.

2. Main result and its proof

For an operator $A \in \mathcal{B}(H)$ and an orthonormal basis $\mathcal{E}$ of H, the symbol $\mathcal{M}_{\mathcal{E}}(A)$ stands for the representation matrix of A with respect to $\mathcal{E}$.

The main result of this paper is the following theorem.

Theorem 2.1. *Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a map. The following assertions are equivalent:*

(i) *For all $A, B \in \mathcal{B}(H)$ and every orthonormal basis $\mathcal{E}$ of H,*

$$\mathcal{M}_{\mathcal{E}}(A \circ B) \text{ is symmetric} \quad \Leftrightarrow \quad \mathcal{M}_{\mathcal{E}}(\Phi(A) \circ \Phi(B)) \text{ is symmetric.}$$

(ii) *For all $A, B \in \mathcal{B}(H)$ and every conjugation C on H,*

$$A \circ B \text{ is } C\text{-symmetric} \quad \Leftrightarrow \quad \Phi(A) \circ \Phi(B) \text{ is } C\text{-symmetric.}$$

(iii) *Φ has one of the following two forms:*

$$A \mapsto f(A)A \quad \text{or} \quad A \mapsto f(A)A^*,$$

where $f : \mathcal{B}(H) \to \mathbb{C} \setminus \{0\}$.

The reverse implication in assertion (ii) of the previous theorem is indispensable, as demonstrated by the following example.

Example 2.2. For $t > 0$, consider the orthogonal sum $A_t = B_t \oplus 0 \in \mathcal{B}(H)$, where B_t is the operator defined with respect to an orthonormal basis by the following matrix:

$$B_t = \begin{array}{c} \begin{bmatrix} 1 & t & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} .$$

As the trace of the operator $B_t^* B_t^2 B_t^{*2} B_t - B_t B_t^{*2} B_t^2 B_t^*$ is equal to $t^2 > 0$, we obtain by [7, Proposition 2.5] that B_t is not C-symmetric for any conjugation C, and hence so is the operator A_t by [8, Lemma 1]. Therefore, letting $\Omega = \{A_t : t > 0\}$, we obtain that any map Φ vanishing on $\mathcal{B}(H)\backslash\Omega$ must satisfy the direct implication of assertion (ii) of the previous theorem for every conjugation. Indeed, if $T = A_t$ and $S = A_s$ for some $t, s > 0$, then we have $T \circ S = 2A_{t+s}$ is never complex symmetric; on the other hand, if $T \notin \Omega$ or $S \notin \Omega$, then $\Phi(T) \circ \Phi(S) = 0$ is C-symmetric for all conjugations C.

Note that the equivalence (i)⇔(ii) of Theorem 2.1 follows easily by (1.1), and that the implication (iii)⇒(ii) is trivial. So in order to prove the theorem, we need only to show that (ii)⇒(iii). In what follows, Φ shall denote a map on $\mathcal{B}(H)$ that satisfies the second assertion of Theorem 2.1.

A bounded linear operator is called *diagonal* if it has a diagonal matrix representation with respect to some orthonormal basis. From [2, Lemma 1], we recall that an operator is diagonal with respect to an orthonormal basis $\{e_i\}$ if and only if it is C_i-symmetric with respect to the conjugations given by

$$C_i e_j = (-1)^{\delta_{ij}} e_j \quad \text{for all } j \geq 1, \tag{2.1}$$

where i varies in $\mathbb{N}$.

Remark 2.3. Let D, T be two bounded linear operators acting on a complex separable Hilbert space K such that

$$D \text{ is } C\text{-symmetric} \quad \Rightarrow \quad T \text{ is } C\text{-symmetric}$$

for every conjugation C on K. If D is diagonal with respect to an orthonormal basis $\{e_i\}$, then so is T by (2.1); furthermore, it follows by the proof of [2, Lemma 3] that

$$\langle De_n, e_n \rangle = \langle De_m, e_m \rangle \quad \Rightarrow \quad \langle Te_n, e_n \rangle = \langle Te_m, e_m \rangle$$

for all $n, m \geq 1$.

As a consequence of the previous remark, for all $A, B \in \mathcal{B}(H)$, we have

$$A \in \mathbb{C}I \quad \Leftrightarrow \quad A \text{ is } C\text{-symmetric for every conjugation } C \text{ on } H \tag{2.2}$$

and

$$A \circ B \in \mathbb{C}I \quad \Leftrightarrow \quad \Phi(A) \circ \Phi(B) \in \mathbb{C}I. \tag{2.3}$$

Lemma 2.4. *Consider an orthogonal decomposition $H = H_1 \oplus H_2$, and let $A_i \in \mathcal{B}(H_i)$ and $T \in \mathcal{B}(H)$, with A_i being a C_i-symmetric operator for $i = 1, 2$. If the implication*

$$A_1 \oplus A_2 \text{ is } C\text{-symmetric} \quad \Rightarrow \quad T \text{ is } C\text{-symmetric}$$

holds for every conjugation C on H, then

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{matrix} H_1 \\ H_2 \end{matrix},$$

where T_i is a C_i-symmetric operator for $i = 1, 2$.

Proof. Obviously, we can assume without loss of generality that H_1 and H_2 are not trivial. It is easy to see that, for $\alpha \in \{1, -1\}$, the map $(\alpha C_1) \oplus C_2$ is a conjugation on H for which the operator $A_1 \oplus A_2$ is complex symmetric, and hence so is T with respect to the same conjugation. Let e and f be vectors in H_1 and H_2, respectively. It follows that

$$\begin{aligned}\langle Te, f\rangle &= \langle [(\alpha C_1) \oplus C_2]\, f, [(\alpha C_1) \oplus C_2]\, Te\rangle \\ &= \langle [(\alpha C_1) \oplus C_2]\, f, T^* \left[(\alpha C_1) \oplus C_2\right] e\rangle \\ &= \langle C_2 f, T^*(\alpha C_1) e\rangle \\ &= \overline{\alpha}\langle C_2 f, T^* C_1 e\rangle,\end{aligned}$$

and similarly, we get that

$$\langle Tf, e\rangle = \alpha \langle C_1 e, T^* C_2 f\rangle.$$

Taking $\alpha = 1$ and $\alpha = -1$, respectively, we obtain

$$\langle Te, f\rangle = \langle C_2 f, T^* C_1 e\rangle = -\langle C_2 f, T^* C_1 e\rangle$$

and

$$\langle Tf, e\rangle = \langle C_1 e, T^* C_2 f\rangle = -\langle C_1 e, T^* C_2 f\rangle.$$

Therefore, $\langle Te, f\rangle = \langle Tf, e\rangle = 0$. Now, since e and f are arbitrary, we infer that T has the form

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{array}{l} H_1 \\ H_2 \end{array}.$$

Finally, from the fact that T is $(C_1 \oplus C_2)$-symmetric, one can readily see that T_1 and T_2 are complex symmetric with respect to C_1 and C_2, respectively. $\square$

For an operator $A \in \mathcal{B}(H)$, we denote by $\mathrm{Ran}(A)$, $\mathrm{Ker}(A)$, and $\mathrm{Rank}(A)$, respectively, the range, the null space, and the rank of A.

It is worth mentioning that if R is an operator whose rank is less than or equal to one, then it can be expressed as an orthogonal sum

$$R = R_0 \oplus 0, \tag{2.4}$$

where R_0 is an operator acting on the at most two-dimensional space $\mathrm{Ran}(R) + \mathrm{Ker}(R)^\perp$. It should also be noted that every operator acting on a space K, with $\dim K \leq 2$, is complex symmetric. Indeed, this is trivial if $\dim K \leq 1$; the other case was proved in [6, Example 6].

Corollary 2.5. *Let $R, T \in \mathcal{B}(H)$ with* $\mathrm{Rank}(R) \leq 1$ *and such that*

$$R \textit{ is } C\textit{-symmetric} \quad \Rightarrow \quad T \textit{ is } C\textit{-symmetric}$$

for every conjugation C on H. Then

$$T = \begin{bmatrix} T_0 & 0 \\ 0 & \lambda I \end{bmatrix} \begin{array}{l} \mathrm{Ran}(R) + \mathrm{Ker}(R)^\perp \\ \left(\mathrm{Ran}(R) + \mathrm{Ker}(R)^\perp\right)^\perp \end{array}$$

for some operator T_0 and $\lambda \in \mathbb{C}$.

Proof. It follows by (2.4) and Lemma 2.4 that T has the form

$$T = \begin{bmatrix} T_0 & 0 \\ 0 & T_1 \end{bmatrix} \begin{array}{l} \operatorname{Ran}(R) + \operatorname{Ker}(R)^\perp \\ \big(\operatorname{Ran}(R) + \operatorname{Ker}(R)^\perp\big)^\perp \end{array},$$

and that the operator T_1 is complex symmetric with respect to all conjugations on $\big(\operatorname{Ran}(R) + \operatorname{Ker}(R)^\perp\big)^\perp$. Therefore, by (2.2), T_1 is a scalar multiple of the identity, and consequently, T has the desired form. □

Let $u, v \in H$ be non-zero. As customary, we denote by $u \otimes v$ the rank-one operator given by $(u \otimes v)(x) = \langle x, v\rangle u$ for all $x \in H$. It is well known that every rank-one operator acting on a Hilbert space has such representation.

The following lemma describes Φ on the set of scalar multiple of rank-one orthogonal projections.

Lemma 2.6. *For every non-zero $\lambda \in \mathbb{C}$ and every unit vector $u \in H$, there is a non-zero $\alpha_{u,\lambda} \in \mathbb{C}$ such that $\Phi(\lambda u \otimes u) = \alpha_{u,\lambda} u \otimes u$.*

Proof. The proof consists of three steps.

Step 1. Let $u \in H$ be a unit vector and $\lambda \in \mathbb{C}$ be non-zero. Clearly, the operator

$$(\lambda u \otimes u) \circ (\lambda u \otimes u) = 2\lambda^2 u \otimes u$$

is diagonal with respect to every orthonormal basis containing u. Since the operators $(\lambda u \otimes u) \circ (\lambda u \otimes u)$ and $\Phi(\lambda u \otimes u) \circ \Phi(\lambda u \otimes u)$ are complex symmetric with respect to the same conjugations, it follows by Remark 2.3 that

$$\Phi(\lambda u \otimes u)^2 = \begin{bmatrix} \gamma_{u,\lambda} & 0 \\ 0 & \beta_{u,\lambda} I \end{bmatrix} \begin{array}{l} u \\ u^\perp \end{array}$$

for some distinct complex numbers $\gamma_{u,\lambda}$ and $\beta_{u,\lambda}$. Hence, as $\Phi(\lambda u \otimes u)$ commutes with $\Phi(\lambda u \otimes u)^2$, it also commutes with the spectral projections of $\Phi(\lambda u \otimes u)^2$. Thus,

$$\Phi(\lambda u \otimes u) = \begin{bmatrix} \alpha_{u,\lambda} & 0 \\ 0 & A_{u,\lambda} \end{bmatrix} \begin{array}{l} u \\ u^\perp \end{array}$$

for some $\alpha_{u,\lambda} \in \mathbb{C}$ and $A_{u,\lambda} \in \mathcal{B}(u^\perp)$ satisfying $\alpha_{u,\lambda}^2 = \gamma_{u,\lambda}$ and $A_{u,\lambda}^2 = \beta_{u,\lambda} I$.

Step 2. Let $u \in H$ be a unit vector and $\lambda \in \mathbb{C}$ be non-zero. We shall prove that if $A_{u,\lambda} \neq 0$, then $A_{u,\lambda} = \mu_{u,\lambda} I$ for some $\mu_{u,\lambda} \in \mathbb{C}$. In light of [1, Lemma 2.8], it suffices to show that every unit vector $v \in u^\perp$ is an eigenvector for $\Phi(\lambda u \otimes u)$. Let $v \in u^\perp$ be a unit vector. As $(\lambda u \otimes u) \circ (v \otimes v) = 0$, we obtain by (2.3) that

$$\Phi(\lambda u \otimes u)\Phi(v \otimes v) + \Phi(v \otimes v)\Phi(\lambda u \otimes u) \in \mathbb{C}I.$$

According to the previous step, we have $\Phi(v \otimes v)v = \alpha_{v,1} v$ for some $\alpha_{v,1} \in \mathbb{C}$; consequently,

$$\Phi(\lambda u \otimes u)\alpha_{v,1} v + \Phi(v \otimes v)\Phi(\lambda u \otimes u)v \in \operatorname{Span}\{v\}.$$

Hence,

$$\left(\alpha_{v,1} I + \Phi(v \otimes v)\right) \Phi(\lambda u \otimes u)v \in \operatorname{Span}\{v\};$$

that is,
$$\begin{array}{@{}c@{}c}\begin{bmatrix}2\alpha_{v,1} & 0\\ 0 & \alpha_{v,1}I+A_{v,1}\end{bmatrix} & \begin{matrix}v\\ v^{\perp}\end{matrix}\end{array}\Phi(\lambda u\otimes u)v\in \operatorname{Span}\{v\}.$$
Note that the operator $\alpha_{v,1}I+A_{v,1}$ is invertible in $\mathcal{B}(v^{\perp})$, because otherwise we get
$$\gamma_{v,1}=\alpha_{v,1}^2\in\sigma(A_{v,1})^2=\sigma(A_{v,1}^2)=\{\beta_{v,1}\},$$
where $\sigma(T)$ denotes the spectrum of T. This contradicts the fact that $\gamma_{v,1}\neq\beta_{v,1}$. Thus
$$\begin{array}{@{}c@{}c}\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix} & \begin{matrix}v\\ v^{\perp}\end{matrix}\end{array}\Phi(\lambda u\otimes u)v=\begin{array}{@{}c@{}c}\begin{bmatrix}0 & 0\\ 0 & (\alpha_{v,1}I+A_{v,1})^{-1}(\alpha_{v,1}I+A_{v,1})\end{bmatrix} & \begin{matrix}v\\ v^{\perp}\end{matrix}\end{array}\Phi(\lambda u\otimes u)v=0,$$
and consequently, $\Phi(\lambda u\otimes u)v\in\operatorname{Span}\{v\}$.

Step 3. Now, fix a unit vector $u\in H$ and a non-zero $\lambda\in\mathbb{C}$, and let us prove that $A_{u,\lambda}=0$. Assume the contrary, and let $v\in u^{\perp}$ be any unit vector. It follows by the previous steps that
$$\Phi(\lambda u\otimes u)=\begin{array}{@{}c@{}c}\begin{bmatrix}\alpha_{u,\lambda} & 0 & 0\\ 0 & \mu_{u,\lambda} & 0\\ 0 & 0 & \mu_{u,\lambda}I\end{bmatrix} & \begin{matrix}u\\ v\\ \{u,v\}^{\perp}\end{matrix}\end{array}$$
and
$$\Phi(v\otimes v)=\begin{array}{@{}c@{}c}\begin{bmatrix}\mu_{v,1} & 0 & 0\\ 0 & \alpha_{v,1} & 0\\ 0 & 0 & \mu_{v,1}I\end{bmatrix} & \begin{matrix}u\\ v\\ \{u,v\}^{\perp}\end{matrix}\end{array}$$
with $\mu_{u,\lambda}\neq 0$ and $\alpha_{v,1}^2\neq\mu_{v,1}^2$. Since $\Phi(\lambda u\otimes u)\circ\Phi(v\otimes v)\in\mathbb{C}I$, one can easily see that $\mu_{u,\lambda}\alpha_{v,1}=\mu_{u,\lambda}\mu_{v,1}$, and consequently $\alpha_{v,1}=\mu_{v,1}$, the desired contradiction. Therefore, $A_{u,\lambda}=0$, and $\Phi(\lambda u\otimes u)=\alpha_{u,\lambda}u\otimes u$ with $\alpha_{u,\lambda}^2\neq\beta_{u,\lambda}=0$ as stated. $\square$

Corollary 2.7. *For every $\lambda\in\mathbb{C}$, the operator $\Phi(\lambda I)$ is a scalar multiple of the identity. Furthermore, $\lambda=0$ if and only if $\Phi(\lambda I)=0$.*

Proof. Fix $\lambda\in\mathbb{C}$. We first establish that $\Phi(\lambda I)$ is a scalar multiple of the identity. Let $u\in H$ be a unit vector. As $(u\otimes u)\circ\lambda I=2\lambda u\otimes u$, it follows by the previous lemma and Remark 2.3 that
$$(u\otimes u)\circ\Phi(\lambda I)=\begin{array}{@{}c@{}c}\begin{bmatrix}\alpha & 0\\ 0 & \beta I\end{bmatrix} & \begin{matrix}u\\ u^{\perp}\end{matrix}\end{array}$$
for some $\alpha,\beta\in\mathbb{C}$. Applying the above operator to u, we get that $\Phi(\lambda I)u\in\operatorname{Span}\{u\}$. Since u is arbitrary, we obtain by [1, Lemma 2.8] that there exists $\alpha_\lambda\in\mathbb{C}$ such that $\Phi(\lambda I)=\alpha_\lambda I$.

Now, according to (2.3), for any fixed unit vector $u\in H$, we have
$$\lambda=0\Leftrightarrow u\otimes u\circ(\lambda I)=0\Leftrightarrow 2\alpha_\lambda\alpha_{u,1}u\otimes u=\Phi(u\otimes u)\circ\Phi(\lambda I)\in\mathbb{C}I$$
with $\alpha_{u,1}\in\mathbb{C}$ being non-zero. Therefore,
$$\lambda=0\Leftrightarrow\alpha_\lambda=0\Leftrightarrow\Phi(\lambda I)=0,$$
the desired equivalence. $\square$

Remark 2.8. It follows from the previous corollary that, for every conjugation C, an operator $A \in \mathcal{B}(H)$ is C-symmetric if and only if $\Phi(A)$ is C-symmetric. In particular, $A \in \mathbb{C}I$ if and only if $\Phi(A) \in \mathbb{C}I$.

Corollary 2.9. *Let $A \in \mathcal{B}(H)$ be an operator that has the form $A = T \oplus 0$ with respect to some non-trivial orthogonal decomposition $H = H_1 \oplus H_2$. Then, with respect to the same decomposition, we have $\Phi(A) = S \oplus 0$ for some $S \in \mathcal{B}(H_1)$.*

Proof. Let u and v be unit vectors in H_1 and H_2, respectively. As $\Phi(v\otimes v)\circ\Phi(A) \in \mathbb{C}I$ because $(v \otimes v) \circ A = 0$, Lemma 2.6 implies that

$$(v \otimes v) \circ \Phi(A) = 0.$$

Applying the above operator to u and v, respectively, we obtain

$$\langle \Phi(A)u, v\rangle v = 0 \quad \text{and} \quad \langle \Phi(A)v, v\rangle v + \Phi(A)v = 0.$$

Consequently, $\langle \Phi(A)u, v\rangle = \langle \Phi(A)v, v\rangle = \langle \Phi(A)v, u\rangle = 0$. Since u and v are arbitrary, one can easily see that $\Phi(A)$ should have the form

$$\Phi(A) = \begin{array}{cc} \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{l} H_1 \\ H_2 \end{array} \end{array},$$

which completes the proof. ∎

Lemma 2.10. *Let u and v be two non-zero orthogonal vectors in H. Then, there is a non-zero $\alpha_{u,v} \in \mathbb{C}$ such that*

$$\Phi(u \otimes v) = \alpha_{u,v} u \otimes v \qquad \text{or} \qquad \Phi(u \otimes v) = \alpha_{u,v} v \otimes u.$$

Proof. Using the previous corollary, we can write

$$\Phi(u \otimes v) = \begin{array}{cc} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \|u\|^{-1}u \\ \|v\|^{-1}v \\ \{u, v\}^{\perp} \end{array} \end{array}$$

for some $a, b, c, d \in \mathbb{C}$. Since $(u\otimes v)\circ(u\otimes v) = 0$, and hence $\Phi(u\otimes v)\circ\Phi(u\otimes v) \in \mathbb{C}I$, we obtain $\Phi(u\otimes v)\circ\Phi(u\otimes v) = 0$, meaning that the operator $\Phi(u\otimes v)$ has a square equal to zero, and so $d = -a$ because the trace of $\Phi(u \otimes v)$ should be zero.

Let $w \in \{u, v\}^{\perp}$ be any non-zero vector. It follows by the previous paragraph that

$$\Phi(w \otimes v) = \begin{array}{cc} \begin{bmatrix} -x & z & 0 \\ y & x & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \|w\|^{-1}w \\ \|v\|^{-1}v \\ \{v, w\}^{\perp} \end{array} \end{array}$$

for some $x, y, z \in \mathbb{C}$. As $\Phi(u \otimes v) \circ \Phi(w \otimes v) \in \mathbb{C}I$ because $(u \otimes v) \circ (w \otimes v) = 0$, we have

$$\begin{bmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & y \\ 0 & z & -x \end{bmatrix} \in \mathbb{C}I_3,$$

where I_3 is the 3×3 identity matrix. Calculations yield

$$\begin{bmatrix} 0 & bx & by \\ cx & -2ax & -ay \\ cz & -az & 0 \end{bmatrix} = 0.$$

Note that $a=b=0$ or $a=c=0$. Indeed, if $a\neq 0$ or $bc\neq 0$, then $x=y=z=0$ and $\Phi(w\otimes v)=0$, and so $w\otimes v\in\mathbb{C}I$ by Remark 2.8, a contradiction. Therefore, we have either $\Phi(u\otimes v)=bu\otimes v$ or $\Phi(u\otimes v)=cv\otimes u$. The fact that $\Phi(u\otimes v)\neq 0$ is ensured by Remark 2.8. □

The following lemma describes Φ on the set of all rank-one nilpotent operators.

Lemma 2.11. *One of the following statements holds:*

$$\Phi(u\otimes v)=\alpha_{u,v}u\otimes v \quad \textit{for all non-zero orthogonal vectors } u,v\in H \tag{2.5}$$

or

$$\Phi(u\otimes v)=\alpha_{u,v}v\otimes u \quad \textit{for all non-zero orthogonal vectors } u,v\in H. \tag{2.6}$$

Proof. In light of the previous lemma, it suffices to show that, for all non-zero vectors u_1,v_1,u_2,v_2 such that $u_1\perp v_1$ and $u_2\perp v_2$, the operators $\Phi(u_1\otimes v_1)$ and $\Phi(u_2\otimes v_2)$ have either the form in (2.5) or the form in (2.6). In other words, we need to prove the following equivalence:

$$\Phi(u_1\otimes v_1)=\alpha_{u_1,v_1}u_1\otimes v_1 \Leftrightarrow \Phi(u_2\otimes v_2)=\alpha_{u_2,v_2}u_2\otimes v_2.$$

The proof is divided into two steps.

Step 1. For non-zero vectors u,v_1,v_2 with $u\in\{v_1,v_2\}^{\perp}$, we have

$$\Phi(u\otimes v_1)=\alpha_{u,v_1}u\otimes v_1 \Rightarrow \Phi(u\otimes v_2)=\alpha_{u,v_2}u\otimes v_2$$

and

$$\Phi(v_1\otimes u)=\alpha_{v_1,u}v_1\otimes u \Rightarrow \Phi(v_2\otimes u)=\alpha_{v_2,u}v_2\otimes u. \tag{2.7}$$

Indeed, if $\Phi(u\otimes v_1)=\alpha_{u,v_1}u\otimes v_1$ and $\Phi(u\otimes v_2)=\alpha_{u,v_2}v_2\otimes u$, we get that

$$\Phi(u\otimes v_1)\circ\Phi(u\otimes v_2)=\alpha_{u,v_1}\alpha_{u,v_2}\langle v_2,v_1\rangle u\otimes u+\alpha_{u,v_1}\alpha_{u,v_2}\|u\|^2 v_2\otimes v_1\notin\mathbb{C}I.$$

This leads to a contradiction because $(u\otimes v_1)\circ(u\otimes v_2)=0$. The implication (2.7) can be obtained in the same manner.

Step 2. Let u_1,u_2,v_1,v_2 be non-zero vectors in H such that $u_1\perp v_1$ and $u_2\perp v_2$, and arbitrarily choose a non-zero vector $w\in\{u_1,u_2\}^{\perp}$. Then, by the previous step, we have

$$\begin{aligned}\Phi(u_1\otimes v_1)=\alpha_{u_1,v_1}u_1\otimes v_1 &\Leftrightarrow \Phi(u_1\otimes w)=\alpha_{u_1,w}u_1\otimes w\\ &\Leftrightarrow \Phi(u_2\otimes w)=\alpha_{u_2,w}u_2\otimes w\\ &\Leftrightarrow \Phi(u_2\otimes v_2)=\alpha_{u_2,v_2}u_2\otimes v_2.\end{aligned}$$

This completes the proof. □

Corollary 2.12. *On the set of rank-one operators, Φ has one of the following forms:*

$$R\mapsto f(R)R \quad \textit{or} \quad R\mapsto f(R)R^*,$$

where $f:\mathcal{B}(H)\mapsto\mathbb{C}\setminus\{0\}$.

Proof. Assume first that (2.5) holds and let us show that, for every rank-one operator $R \in \mathcal{B}(H)$, there is a non-zero $\alpha_R \in \mathbb{C}$ such that $\Phi(R) = \alpha_R R$.

Let $R \in \mathcal{B}(H)$ be a rank-one operator. Then, we can write $R = (au + bv) \otimes v$ where u and v are two orthonormal vectors in H, and $a, b \in \mathbb{C}$. In view of Lemma 2.6 and Lemma 2.11, we can assume that a and b are non-zero. Choose an arbitrary non-zero vector $w \in \{u, v\}^{\perp}$. Using Corollary 2.9, we may write

$$\Phi(R) = (\alpha u + \beta v) \otimes v + (\alpha' u + \beta' v) \otimes u$$

for some $\alpha, \beta, \alpha', \beta' \in \mathbb{C}$. Since $R \circ (u \otimes w) = R \circ \big(w \otimes (-\bar{b}u + \bar{a}v)\big) = 0$, we obtain

$$\Phi(R) \circ \Phi(u \otimes w) \in \mathbb{C}I \quad \text{and} \quad \Phi(R) \circ \Phi\left(w \otimes (-\bar{b}u + \bar{a}v)\right) \in \mathbb{C}I.$$

Hence, it follows by the previous lemma that

$$[(\alpha u + \beta v) \otimes v + (\alpha' u + \beta' v) \otimes u] \circ (u \otimes w) = 0$$

and

$$[(\alpha u + \beta v) \otimes v + (\alpha' u + \beta' v) \otimes u] \circ \big(w \otimes (-\bar{b}u + \bar{a}v)\big) = 0.$$

The first equality implies that $(\alpha' u + \beta' v) \otimes w = 0$, and so $\alpha' u + \beta' v = 0$. The second one yields

$$\begin{aligned}
(-b\alpha + a\beta) w \otimes v &= ((\alpha u + \beta v) \otimes v) \circ \big(w \otimes (-\bar{b}u + \bar{a}v)\big) \\
&= [(\alpha u + \beta v) \otimes v + (\alpha' u + \beta' v) \otimes u] \circ \big(w \otimes (-\bar{b}u + \bar{a}v)\big) \\
&= 0.
\end{aligned}$$

Therefore,

$$\alpha' = \beta' = 0 \quad \text{and} \quad b\alpha = a\beta.$$

Clearly, it follows from the second equality above that if one of α and β is zero, then so is the other, which implies that $\Phi(R) = 0$, and hence R would be a scalar multiple of the identity, a contradiction. Consequently, $\Phi(R) = b^{-1}\beta R$.

Assume now that (2.6) holds. Then, it is easy to see that the map $\Psi(X) = \Phi(X^*)$ satisfies the second assertion of Theorem 2.1 and the form (2.5) in Lemma 2.11. Hence, we obtain that, for every rank-one operator $R \in \mathcal{B}(H)$,

$$\Phi(R) = \Phi((R^*)^*) = \Psi(R^*) = \alpha_R R^*$$

for some non-zero $\alpha_R \in \mathbb{C}$. This ends the proof of the corollary. $\square$

In the remainder of this section, we assume that Φ satisfies (2.5) and we aim to show that Φ has the first form in assertion (iii) of Theorem 2.1. The case where Φ satisfies (2.6) can be treated by considering the map Ψ defined in the previous proof, and therefore, in this case, Φ would have the second form in assertion (iii) of Theorem 2.1.

Lemma 2.13. *Let $A \in \mathcal{B}(H)$. Then, A and $\Phi(A)$ have a mutual eigenvector.*

Proof. The proof is divided into two steps:

Step 1. Firstly, we assume that A has only one eigenvalue where the associated eigenspace is spanned by $u \in H$. We distinguish two cases:

Case 1. $Au = 0$. Let $v \in H$ be a non-zero vector such that $A^*v = 0$. We have $\Phi(A) \circ \Phi(u \otimes v) \in \mathbb{C}I$ because $A \circ (u \otimes v) = 0$, and since $\Phi(u \otimes v)$ is rank-one by the previous corollary, we get

$$\Phi(A)(u \otimes v) + (u \otimes v)\Phi(A) = 0.$$

Hence, $(\Phi(A)u) \otimes v = -u \otimes (\Phi(A)^*v)$, and consequently $\Phi(A)u \in \operatorname{Span}\{u\}$.

Case 2. $Au = \alpha u$ with $\alpha \neq 0$. Since the spectrum of A^* contains only $\overline{\alpha}$, then the operator $\overline{\alpha}I + A^*$ is invertible. Let $h \in H$ be a vector such that $u = \overline{\alpha}h + A^*h$. Since

$$A \circ (u \otimes h) = \alpha u \otimes h + u \otimes (A^*h) = u \otimes (\overline{\alpha}h + A^*h) = u \otimes u,$$

we get by Corollary 2.5 and the previous corollary that

$$T := (\Phi(A)u) \otimes h + u \otimes (\Phi(A)^*h) = \Phi(A) \circ (u \otimes h) = \begin{array}{c} \begin{bmatrix} \gamma & 0 \\ 0 & \lambda I \end{bmatrix} \end{array} \begin{array}{l} \|u\|^{-1}u \\ \{u\}^{\perp} \end{array},$$

where $\gamma, \lambda \in \mathbb{C}$. Note that, since T has at most rank two, at least one of γ and λ is zero. Furthermore, they cannot both be zero because otherwise the operator $A \circ (u \otimes h)$ would be a scalar multiple of the identity.

If $\lambda \neq 0$, we obtain that T is a rank-two operator, which implies that

$$u \in \operatorname{Ran}\left((\Phi(A)u) \otimes h + u \otimes (\Phi(A)^*u)\right) = \{u\}^{\perp},$$

yielding a contradiction. Therefore, $\lambda = 0$, and consequently,

$$\|h\|^2\Phi(A)u + \langle h, \Phi(A)^*h\rangle u = Th \in \operatorname{Span}\{u\},$$

and so $\Phi(A)u \in \operatorname{Span}\{u\}$.

Step 2. Suppose now that there are two linearly independent vectors u_1 and u_2 such that $Au_i = \alpha_i u_i$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$, and let v_1 and v_2 be two linearly independent unit vectors in H such that $A^*v_i = \overline{\alpha_i}v_i$. Then

$$A \circ (u_i \otimes v_j) = (\alpha_i + \alpha_j)u_i \otimes v_j \quad \text{for } 1 \leq i, j \leq 2.$$

Using Corollary 2.5 and the previous corollary, we get

$$\Phi(A) \circ (u_i \otimes v_j) = \begin{bmatrix} T_{i,j} & 0 \\ 0 & \lambda_{i,j}I \end{bmatrix} \begin{array}{l} \operatorname{Span}\{u_i, v_j\} \\ \{u_i, v_j\}^{\perp} \end{array},$$

where $T_{i,j} \in \mathcal{B}(\operatorname{Span}\{u_i, v_j\})$ and $\lambda_{i,j} \in \mathbb{C}$ for $1 \leq i, j \leq 2$. Letting $w_{i,j}$ be any non-zero vector in $\{u_i, v_j\}^{\perp}$, we obtain

$$\begin{aligned} \lambda_{i,j}w_{i,j} = \begin{bmatrix} T_{i,j} & 0 \\ 0 & \lambda_{i,j}I \end{bmatrix} \begin{array}{l} \operatorname{Span}\{u_i, v_j\} \\ \{u_i, v_j\}^{\perp} \end{array} w_{i,j} &= \left(\Phi(A) \circ (u_i \otimes v_j)\right) w_{i,j} \\ &= \left((\Phi(A)u_i) \otimes v_j + u_i \otimes (\Phi(A)^*v_i)\right) w_{i,j} \\ &= \langle w_{i,j}, \Phi(A)^*v_j\rangle u_i, \end{aligned}$$

which implies that $\lambda_{i,j} = 0$. Consequently,

$$\operatorname{Ran}\left(\Phi(A) \circ (u_i \otimes v_j)\right) \subset \operatorname{Span}\{u_i, v_j\} \quad \text{for } 1 \leq i, j \leq 2. \tag{2.8}$$

It follows that, for all $1 \leq i, j \leq 2$,

$$\Phi(A)u_i + \langle v_j, \Phi(A)^*v_j\rangle u_i = \left(\Phi(A) \circ (u_i \otimes v_j)\right) v_j \in \operatorname{Span}\{u_i, v_j\},$$

and hence $\Phi(A)u_i \in \operatorname{Span}\{u_i, v_j\}$. If we suppose that

$$\operatorname{Span}\{u_i, v_1\} \cap \operatorname{Span}\{u_i, v_2\} = \operatorname{Span}\{u_i\} \quad \text{for some } i \in \{1,2\},$$

we readily see that the u_i is an eigenvector for $\Phi(A)$. Assume that

$$\operatorname{Span}\{u_i, v_1\} = \operatorname{Span}\{u_i, v_2\} \quad \text{for } i = 1,2.$$

Then, $v_1 = \alpha u_1 + \beta v_2$ and $v_2 = \alpha' u_2 + \beta' v_1$ for some $\alpha, \alpha', \beta, \beta' \in \mathbb{C}$. Hence, $(1 - \beta\beta')v_1 = \alpha u_1 + \alpha'\beta u_2$. Obviously, $1 - \beta\beta' \neq 0$ because $\alpha \neq 0$ and u_1, u_2 are linearly independent. Consequently, $v_1 \in \operatorname{Span}\{u_1, u_2\}$. In a similar way, we show that $v_2 \in \operatorname{Span}\{u_1, u_2\}$. Hence, $\operatorname{Span}\{u_1, u_2\} = \operatorname{Span}\{v_1, v_2\}$, and so A has the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{array}{l} \operatorname{Span}\{u_1, u_2\} \\ \{u_1, u_2\}^{\perp} \end{array} .$$

Let w be an eigenvector of A_2^*. Then, we can show as in (2.8) that

$$\operatorname{Ran}\left(\Phi(A) \circ (u_1 \otimes w)\right) \subset \operatorname{Span}\{u_1, w\},$$

and so, $\Phi(A)u_1 \in \operatorname{Span}\{u_1, w\}$. Since $\Phi(A)u_1 \in \operatorname{Span}\{u_1, u_2\}$ and $w \in \{u_1, u_2\}^{\perp}$, we obtain necessarily that $\Phi(A)u_1 \in \operatorname{Span}\{u_1\}$, which ends the proof of the lemma. □

Recall that a rank-one operator $x \otimes y$ is C-symmetric for some conjugation C if and only if $Cx \in \operatorname{Span}\{y\}$ (or equivalently, $Cy \in \operatorname{Span}\{x\}$), see [6, Lemma 2]. Then, taking into account the well-known fact that every rank-one operator is complex symmetric, we obtain that, for all non-zero vectors $x, y \in H$, there exists a conjugation C on H such that $Cy \in \operatorname{Span}\{x\}$.

It is worth mentioning that $C(x \otimes y)C = Cx \otimes Cy$ for all conjugations C and vectors $x, y \in H$.

With these results at hand, we are ready to prove the main result of this paper.

Proof of the implication (ii)$\Rightarrow$(iii) *in Theorem 2.1.* Fix an operator $A \in \mathcal{B}(H)$, and let us show that $\Phi(A) = \lambda A$ for some non-zero $\lambda \in \mathbb{C}$. First note that by Corollary 2.7, we can assume that A is not a scalar multiple of the identity. The previous lemma ensures the existence of a non-zero vector $u \in H$ satisfying $Au = \alpha u$ and $\Phi(A)u = \beta u$ for some $\alpha, \beta \in \mathbb{C}$. Let $h \in H$ be a non-zero vector. It follows by the previous remark that the operator

$$A \circ (u \otimes h) = u \otimes (\overline{\alpha} h + A^* h)$$

is C-symmetric for some conjugation C satisfying $C(\overline{\alpha} h + A^* h) \in \operatorname{Span}\{u\}$, and hence so is the operator

$$u \otimes (\overline{\beta} h + \Phi(A)^* h) = f(u \otimes h)^{-1} \Phi(A) \circ \Phi(u \otimes h),$$

where f is the map obtained in Corollary 2.12. Notice that

$$\begin{aligned} \overline{\alpha} h + A^* h = 0 &\Leftrightarrow A \circ (u \otimes h) = 0 \Leftrightarrow \Phi(A) \circ \Phi(u \otimes h) \in \mathbb{C}I \\ &\Leftrightarrow \Phi(A) \circ (u \otimes h) = 0 \Leftrightarrow \overline{\beta} h + \Phi(A)^* h = 0. \end{aligned}$$

In the case $\overline{\alpha} h + A^* h \neq 0$, we obtain by the previous remark that

$$\operatorname{Span}\{\overline{\alpha} h + A^* h\} = C \operatorname{Span}\{u\} = \operatorname{Span}\{\overline{\beta} h + \Phi(A)^* h\}.$$

In both cases, we can find a non-zero $\lambda_h \in \mathbb{C}$ such that

$$\overline{\beta}h + \Phi(A)^*h = \lambda_h(\overline{\alpha}h + A^*h).$$

Since h is arbitrary, we get the existence of a non-zero $\lambda \in \mathbb{C}$ such that $\overline{\beta}I+\Phi(A)^* = \overline{\lambda}(\overline{\alpha}I + A^*)$, and consequently $\Phi(A) = \lambda A + \gamma I$ for some $\gamma \in \mathbb{C}$.

Finally, it remains to show that $\gamma = 0$. Let $x \in H$ such that x and A^*x are linearly independent, and let J be a conjugation on H that satisfies $J(\overline{\alpha}x+A^*x) \in \operatorname{Span}\{u\}$. Then,

$$\begin{aligned} u \otimes (\overline{\alpha}x + A^*x) \text{ is } J\text{-symmetric} &\Rightarrow A \circ (u \otimes x) \text{ is } J\text{-symmetric} \\ &\Rightarrow \Phi(A) \circ \Phi(u \otimes x) \text{ is } J\text{-symmetric} \\ &\Rightarrow \lambda A \circ (u \otimes x) + 2\gamma u \otimes x \text{ is } J\text{-symmetric} \\ &\Rightarrow (\gamma u) \otimes x \text{ is } J\text{-symmetric.} \end{aligned}$$

So if γ is non-zero, we obtain that

$$(\overline{\alpha}x + A^*x) \in J\operatorname{Span}\{u\} = J\operatorname{Span}\{\gamma u\} = \operatorname{Span}\{x\},$$

which contradicts the fact that x and A^*x are linearly independent. This ends the proof of the theorem. ☐

We conclude this section with the following questions:

Question 2.14. Fix a conjugation C on H. Does every map on $\mathcal{B}(H)$ that preserves the Jordan product of C-symmetric operators, in both directions, have one of the following forms:

$$A \mapsto f(A)UAU^{-1} \quad \text{or} \quad A \mapsto f(A)UA^*U^{-1},$$

where U is a unitary (or anti unitary) operator commuting with C and $f : \mathcal{B}(H) \to \mathbb{C} \setminus \{0\}$?

Question 2.15. Does Theorem 2.1 remain valid in the setting of infinite-dimensional Hilbert spaces?

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